CE541A DYNAMICS OF STRUCTURES

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1. Single-Degree-of-Freedom Systems

Glossary:

SDOF: single degree of freedom
MDOF: multiple degree of freedom

Motivation for studying SDOF systems:

- Some systems are nearly 1DOF;
- First mode is dominant;
- By model decoupling,

$$\text{MDOF} \iff \sum_i \text{SDOF}_i$$

1.1. Integral transformation.

Some problems are easier to do in transform domain.

Common integral transformations:

- Laplace transformation;
- Inverse Laplace transformation;
- Convolution integral.

Generalized vibration problem:

Excitation $F(t)$	→	System characteristic $G(t)$	→	Response $x(t)$

Here $G(t)$ is a differential operator:

$$G(t) \equiv m\frac{\mathrm{d}^2}{\mathrm{d}t^2} + c\frac{\mathrm{d}}{\mathrm{d}t} + k$$

, and they are interrelated as:

$$G(t)x(t) = F(t)$$

Generalized vibration problem after integral transformation:

Transformed excitation $\bar{F}(s)$	→	Transfer function $G(s)$	→	Transformed response $\bar{x}(s)$

Here $G(s)$ is an algebraic expression:

$$G(s) \equiv \frac{1}{ms^2 + cs + k}$$

Date: October 15, 2012.

and

$$\bar{x}(s) \equiv \mathcal{L}\{x(t)\} = \int_0^\infty e^{-st} x(t)\ \mathrm{d}t$$

$$\bar{F}(s) \equiv \mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\ \mathrm{d}t$$

They are interrelated as[1]:

$$\bar{x}(s) = G(s)\bar{F}(s) \tag{1}$$

Definition 1. ***Generalized impedance of system*** *is*

$$Z(s) \equiv \frac{\bar{F}(s)}{\bar{x}(s)}$$

Definition 2. ***Admittance of system*** *is*

$$Y(s) \equiv \frac{\bar{x}(s)}{\bar{F}(s)}$$

1.2. Classification of Structural Dynamics Problems.

Classification	$\bar{F}(s)$	$G(s)$	$\bar{x}(s)$
Analysis	$\surd$	$\surd$	?
Instrumentation	?	$\surd$	$\surd$
Synthesis/Identification	$\surd$	?	$\surd$

Analysis: given excitation and system, determine response.

Instrumentation: response and system characteristics known, find excitation.

Synthesis/Identification: given the excitation and response, determine system characteristics. (Solution is not unique.)
Application:
- Optimum design
- Structural health monitoring, a.k.a. SHM

1.3. Indicial response and impulsive response.

Definition 3. ***Indicial response*** $g(t)$ *is the response of a system with zero initial condition (I.C.) to a unit step function* $u(t)$ *applied at* $t = 0$.

From equation 1, we have

$$\mathcal{L}\{g(t)\} = G(s)\mathcal{L}\{u(t)\}$$

Since $\mathcal{L}\{u(t)\} = \frac{1}{s}$, indicial response is

$$g(t) = \mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\} \tag{2}$$

Definition 4. ***Impulsive response*** $h(t)$ *is the response of a system at rest to a unit impulse* $\delta(t)$ *applied at* $t = 0$.

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[1] $G(s)$ represents steady-state response per unit sinusoidal input as a function of excitation frequency.

Since $\mathcal{L}\{\delta(t)\} = 1$, impulsive response is

$$h(t) = L^{-1}\{G(s)\} \tag{3}$$

Relationship between indicial and impulsive response

Since

$$\begin{aligned} \mathcal{L}\left\{\frac{\mathrm{d}g(t)}{\mathrm{d}t}\right\} &= s\bar{g}(s) - g(0) \\ &= G(s) - g(0) \\ &= \mathcal{L}\{h(t)\} - g(0) \end{aligned}$$

, hence

$$h(t) = \frac{\mathrm{d}g(t)}{\mathrm{d}t} + g(0)\delta(t) \tag{4}$$

That means, displacement response of a linear system to a unit impulse **equals to** its velocity response to a unit step load.

Indicial response of a linear damped SDOF system

The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 1 \quad (t \geq 0)$$

, with I.C.

$$x(0) = \dot{x}(0) = 0$$

General solution to this equation is

$$x(t) = Ce^{\frac{-c}{2m}t}\cos(\omega_d t - \alpha) + \frac{1}{k}$$

Using I.C.,

$$\begin{cases} C\cos\alpha + \frac{1}{k} = 0 \\ C\left(\frac{-c}{2m}\cos\alpha + \omega_d \sin\alpha\right) = 0 \end{cases}$$

Let $\zeta = \sin\alpha$, we have

$$\zeta = \frac{c}{2\sqrt{mk}} \tag{5}$$

$$C = -\frac{1}{k\sqrt{1-\zeta^2}} \tag{6}$$

and

$$\omega = \sqrt{\frac{k}{m}} \tag{7}$$

$$\omega_d = \sqrt{1-\zeta^2}\ \omega \tag{8}$$

So, the indicial response is

$$g(t) = \frac{1}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{c}{2m}t}\cos(\omega_d t - \alpha)\right] \tag{9}$$

$$g'(t) = \frac{\omega}{k\sqrt{1-\zeta^2}}e^{-\frac{c}{2m}t}\sin(\omega_d t) \tag{10}$$

Impulsive response of a linear damped SDOF system

The equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (t \geq 0)$$

, with I.C.

$$\begin{cases} x(0) = 0 \\ \dot{x}(0) = \frac{1}{m} \end{cases}$$

General solution to this equation is

$$x(t) = Ce^{\frac{-c}{2m}t}\cos(\omega_d t - \alpha)$$

, where

$$\omega_d = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

Using I.C.,

$$\begin{cases} C\cos\alpha = 0 \\ C\left(\frac{-c}{2m}\cos\alpha + \omega_d \sin\alpha\right) = \frac{1}{m} \end{cases}$$

we have

$$\begin{cases} \alpha = \frac{\pi}{2} \\ C = \frac{1}{m\omega_d} \end{cases}$$

The impulsive response is

$$h(t) = \frac{\omega}{k\sqrt{1-\zeta^2}} e^{-\frac{c}{2m}t} \sin(\omega_d t) \tag{11}$$

We can see that

$$h(t) = g'(t)$$

1.4. **Duhamels Integral (convolution integral).** Suppose the indicial response of a linear system is $g(t)$, regard a random excitation $f(t)$ as a superposition of infinitely many step functions. The differential change in response caused by the step function applied at time τ is

$$\mathrm{d}x(t) = \mathrm{d}f(\tau)\ g(t-\tau)$$

So the response is

$$x(t) = f(0)g(t) + \int_0^t f'(\tau)g(t-\tau)\ \mathrm{d}\tau$$

Isn't $g(0) = 0$?

Through integration by parts,

$$x(t) = f(t)g(0) + \int_0^t g'(t-\tau)f(\tau)\ \mathrm{d}\tau$$

Using differentiation under the integral sign,

$$x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t g(t-\tau)f(\tau)\ \mathrm{d}\tau \tag{12}$$

Or from equation 4, we get[2]

$$x(t) = \int_0^t h(t-\tau) f(\tau)\ \mathrm{d}\tau \tag{13}$$

Limitations of convolution integral:

- System must be linear;
- Excitation $f(t)$ defined only for $t > 0$.

1.5. **General solution for forced vibration of a damped system.** Using convolution integral and impulsive response of a linear damped SDOF system, the solution with zero I.C. is

$$x(t) = \int_0^t \frac{\omega}{k\sqrt{1-\zeta^2}} e^{-\frac{c}{2m}(t-\tau)} \sin\omega_d(t-\tau)\ F(\tau)\ \mathrm{d}\tau$$

With I.C. $x(0) = x_0, \dot{x}(0) = \dot{x}_0$, the complementary solution is

$$x(t) = e^{-\frac{c}{2m}t}(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega x_0}{\omega_d} \sin\omega_d t)$$

The complete solution then becomes:

$$x(t) = e^{-\frac{c}{2m}t}(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega x_0}{\omega_d} \sin\omega_d t) + \int_0^t \frac{\omega}{k\sqrt{1-\zeta^2}} e^{-\frac{c}{2m}(t-\tau)} \sin\omega_d(t-\tau)\ F(\tau)\ \mathrm{d}\tau \tag{14}$$

1.6. **Complex number representation of SDOF.** Suppose the equation of motion is:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos\omega t \tag{15}$$

Using knowledge of complex number, it can be rewritten as:

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \tag{16}$$

, with I.C.

$$\begin{cases} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{cases}$$

Suppose a particular solution is $ae^{i\omega t}$, then write the general solution as superposition of homogeneous solution and the particular solution, we have:

$$x(t) = \lambda_1 e^{s_1 t} + \lambda_2 e^{s_2 t} + ae^{i\omega t} \tag{17}$$

Definition 5. *Frequency ratio*

$$r \equiv \frac{\omega}{\omega_n}$$

The excited amplitude is

$$a = \frac{F_0}{k} \frac{1}{(1-r^2) + i(2\zeta r)} = \frac{F_0}{k} \left|H(r,\zeta)\right| e^{-i\varphi} \tag{18}$$

, where

$$\left|H(r,\zeta)\right| \equiv \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

[2]This formula can also be derived from the other point of view: regarding excitation as a superposition of infinitely many impulse.

$$\varphi \equiv \arctan\left(\frac{2\zeta r}{1-r^2}\right)$$

Define

$$z(s) = ms^2 + cs + k \tag{19}$$

Let $z(s) = 0$, we get the roots as:

$$s_{1,2} = \begin{cases} -\zeta\omega \pm i\sqrt{1-\zeta^2}\omega & (0 < \zeta < 1) \\ -\zeta\omega \pm \sqrt{\zeta^2 - 1}\omega & (\zeta > 1) \end{cases} \tag{20}$$

Their respective complex amplitude is

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ s_1 & s_2 \end{pmatrix}^{-1} \begin{pmatrix} x_0 - a \\ \dot{x}_0 - i\omega a \end{pmatrix} \tag{21}$$

The following are some terms used in "modal analysis".

Definition 6. ***Receptance*** *is the ratio of steady state displacement to excitation.*[3]

$$\left[\frac{x}{f}\right] \equiv \frac{\textit{ss disp}}{\textit{excitation}}$$

Definition 7. *A* ***Nyquist plot*** *is a parametric plot of receptance* $[\frac{x}{f}]$ *with respect to frequency ratio* r*.*

Definition 8. *For light damped system,* $\max\left|\frac{x}{f}\right|$ *occurs at* $r = \sqrt{1-2\zeta^2}$*. Then frequency ratios* r_1, r_2 *that correspond to* $\frac{1}{\sqrt{2}}\max\left|\frac{x}{f}\right|$ *are called (lower and upper)* ***half power points****.*

Definition 9. ***Bandwidth*** *is defined as the frequency difference between the lower and upper half power points.*

$$\Delta\omega \equiv \omega_2 - \omega_1$$

1.7. **Application of Complex Number Representation.** For a vehicle traveling on a rough road.

Height of road is:

$$y(x) = A\sin\frac{2\pi x}{L}$$

Horizontal position of vehicle is:

$$x(t) = vt$$

Then

$$y(t) = A\sin\Omega t$$

, with

$$\Omega \equiv \frac{2\pi v}{L}$$

Denote z as the vertical position of vehicle, the equation of motion is:

$$m\ddot{z} = -k(z-y) - c(\dot{z} - \dot{y})$$

[3]Note that in our case,

$$\left[\frac{x}{f}\right] = \frac{a}{F_0} = \frac{1}{z(i\omega)}$$

Write $y(t)$ as

$$y(t) = Ae^{i\Omega t}$$

Then excitation can be written as

$$\begin{aligned} f(t) &= ky + c\dot{y} \\ &= A(k + i\Omega c)e^{i\Omega t} \\ &= Ak[1 + i(2\zeta r)]e^{i\Omega t} \\ &= F_0 e^{i(\Omega t+\alpha)} \end{aligned}$$

, with

$$\begin{cases} \alpha \equiv \arctan(2\zeta r) \\ F_0 \equiv Ak\sqrt{1 + (2\zeta r)^2} \end{cases}$$

Now the equation of motion can be written as:

$$m\ddot{z} + c\dot{z} + kz = F_0 e^{i(\Omega t+\alpha)}$$

The steady state solution of the equation is

$$\begin{aligned} z_{ss}(t) &= \frac{F_0}{k} \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} e^{i(\Omega t+\alpha-\varphi)} \\ &= z_p e^{i(\Omega t+\alpha-\varphi)} \end{aligned}$$

, with peak steady state motion

$$z_p \equiv \frac{A\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

Force transmitted to vehicle is

$$\begin{aligned} F_T &\equiv -m\ddot{z} \\ &= (m\Omega^2 z_p)e^{i(\Omega t+\alpha-\varphi)} \end{aligned}$$

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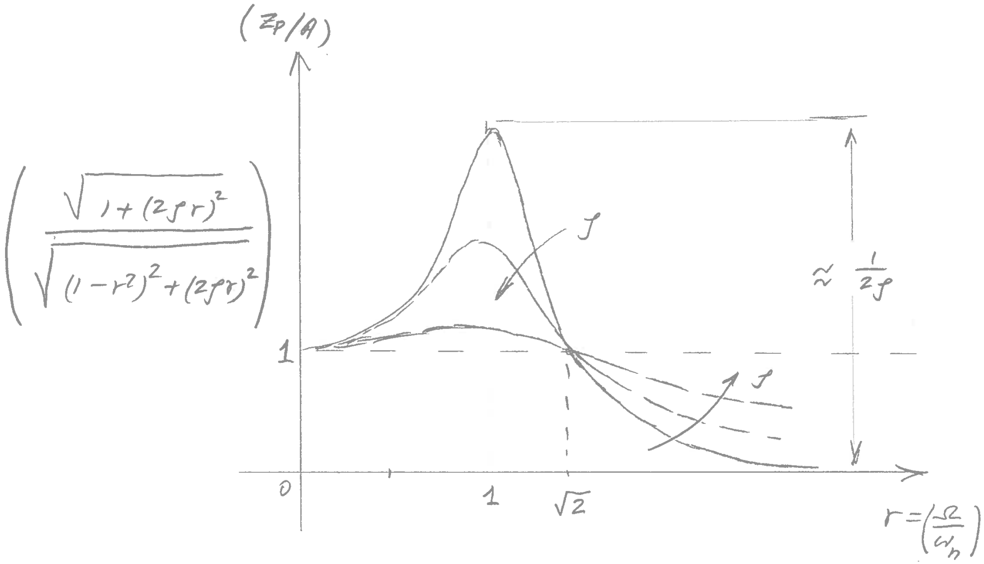


FIGURE 1. Peak Steady State Motion Relative to Input Amplitude

Response to general periodic excitation:

$$p(t + jT_0) = p(t) \qquad j \in \mathbb{Z}.$$

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\omega_0 t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_0 t) \qquad \omega_0 = \frac{2\pi}{T_0}.$$

$$\begin{cases} a_0 = \frac{1}{T_0} \int_0^{T_0} p(t)\,dt \\ a_j = \frac{2}{T_0} \int_0^{T_0} p(t) \cos(j\omega_0 t)\,dt \\ b_j = \frac{2}{T_0} \int_0^{T_0} p(t) \sin(j\omega_0 t)\,dt. \end{cases}$$

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

$$x_{ss} = \frac{F_0}{k} \cdot \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \underbrace{e^{i(\omega t - \varphi)}}_{\text{take real part}}, \quad \varphi = \arctan \frac{2\zeta r}{1-r^2}$$

$$m\ddot{x} + c\dot{x} + kx = p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos(j\omega_0 t) + \sum_{j=1}^{\infty} b_j \sin(j\omega_0 t)$$

$$x_{ss} = \frac{a_0}{k} + \sum_{j=1}^{\infty} \frac{a_j}{k} \frac{1}{\sqrt{(1-r_j^2)^2 + (2\zeta r_j)^2}} \cos(\omega t - \varphi_j)$$

$$+ \sum_{j=1}^{\infty} \frac{b_j}{k} \frac{1}{\sqrt{(1-r_j^2)^2 + (2\zeta r_j)^2}} \sin(\omega t - \varphi_j)$$

$$\left(\varphi_j = \arctan \frac{2\zeta r_j}{1-r_j^2}, \quad r_j = \frac{j\omega_0}{\omega} = j r_1 \right)$$

Shock spectrum:

- The severity of the shock is generally measured in terms of the maximum value of the response.
- Def: The plot of the peak response of a mass-spring system to a given shock as a function of the natural frequency of the system is known as <u>shock spectrum</u>, or <u>response spectrum</u>.

Half-sine pulse: (undamped)

$$F(t) = \begin{cases} F_0 \sin \omega t & (0<t<T,\ T=\frac{\pi}{\omega}) \\ 0 & (t>T,\ t<0) \end{cases}$$

$$= F_0 [\sin \omega t \cdot u(t) + \sin \omega (t-T) \cdot u(t-T)]$$

$$x(t) = \frac{F_0}{k} \cdot \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2} \left\{ \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t\right) \cdot u(t) + \left[\sin \omega(t-T) - \frac{\omega}{\omega_n} \sin \omega_n (t-T)\right] \cdot u(t-T) \right\}$$

$$\frac{|x_{max}| \cdot k}{F_0} = \max \left\{ \frac{2\left(\frac{\omega_n}{\omega}\right)}{1-\left(\frac{\omega_n}{\omega}\right)^2} \cos \frac{\pi}{2} \cdot \left(\frac{\omega_n}{\omega}\right),\ \frac{\left(\frac{\omega_n}{\omega}\right)}{\left(\frac{\omega_n}{\omega}\right)-1} \cdot \sin \frac{2i\pi}{1+\frac{\omega_n}{\omega}} \quad \left(\frac{\omega_n}{\omega} > 2i-1,\ i=1,2,3,\cdots\right) \right\}.$$

Rectangular pulse (undamped):

$$F(t) = \frac{F_0}{2}\left[u(t) - u(t-t_d)\right]$$

$$x(t) = \frac{F_0}{2k}\left\{(1-\cos\omega_n t)\,u(t) - \left[1-\cos\omega_n(t-t_d)\right]u(t-t_d)\right\}$$

$$\frac{|X_{max}|k}{F_0} = \begin{cases} \sin\pi\frac{t_d}{T_n}, & 0<\frac{t_d}{T_n}<\frac{1}{2} \\ 1, & \frac{t_d}{T_n}>\frac{1}{2} \end{cases}$$

Triangular pulse (undamped):

It is too difficult to derive the shock spectrum by hand.

Work-energy relations

$$E_V = \frac{1}{2} k x^2$$

$$E_K = \frac{1}{2} m \dot{x}^2$$

$$F(t) = F_0 \cos \omega t$$

$$X_{ss}(t) = A \cos(\omega t - \alpha)$$

$$W_{ext} = \int_0^T F(t)\, \dot{x}(t)\, dt = \pi F_0 A \sin\alpha$$

$$W_d = \int_0^T -c\,\dot{x}(t) \cdot \dot{x}(t)\, dt = -\pi c \omega A^2$$

In steady state,

$$W_{ext} + W_d = 0$$

$$\Rightarrow \frac{A}{F_0} = \frac{\sin\alpha}{c\omega}$$

$A \approx \frac{F_0}{c\omega}$ why?

Damping loss factor $\equiv \dfrac{|W_d|}{E_{V_{max}}} = \dfrac{\pi c \omega A^2}{\frac{1}{2} k A^2} = \dfrac{2\pi c}{k} \omega$

Equivalent viscous damping:

For a nonlinear damped oscillating system, equivalent viscous damping is:

$$C_{eq} \equiv \frac{|W_{d_{NL}}|}{\pi \omega A^2}$$

so that

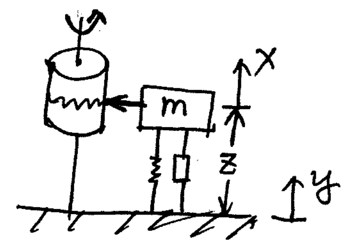
$$|W_{d_{NL}}| = \pi C_{eq} \omega A^2$$

Can we substitute c with C_{eq} in

$$A = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \; ?$$

Note: Coulomb damping is not effective in limiting the response amplitude at resonance. (How to interpret this?)

Instrumentation



$$y(t) = y_0 \cos \omega t$$

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

$$= m\omega^2 y_0 \cos \omega t$$

$$z_{ss}(t) = \frac{m\omega^2 y_0}{k} |H(r,\zeta)| \cdot \cos(\omega t - \varphi) \quad \left(\varphi = \arctan \frac{2\zeta r}{1-r^2} \right)$$

$$= y_0 r^2 |H(r,\zeta)| \cos(\omega t - \varphi)$$

$$\frac{z_{ss\,peak}}{y_0} = r^2 |H(r,\zeta)| = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

(a) If $r \to \infty$, $\frac{z_{ss\,peak}}{y_0} \approx 1$

(b) If $r \to 0$, $\frac{z_{ss\,peak}}{y_0} \approx r^2$

"Optimal damping"

If $\zeta = \zeta_{opt} \approx 0.70$, then $\varphi \approx \frac{\pi}{2} r$.

For $y(t) = \sum_i y_i \sin \omega_i t$, we have

$$z(t) = \sum_i y_i r_i^2 |H(r_i, \zeta)| \sin \omega_i \left(t - \frac{\pi}{2\omega_n}\right)$$

It has a "pure delay" of $t_d = \frac{\pi}{2\omega_n}$.

Error of accelerometer :

$$e \equiv \frac{z_{ss\,peak}\,\omega_n^2 - y_0 \omega^2}{y_0 \omega^2} = |H(r,\zeta)| - 1$$

when $\zeta < \frac{1}{\sqrt{2}} \approx 0.707$, $e_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$, at $r_{max} = \sqrt{1-2\zeta^2}$

If we require $e_{max} \le e$, we have $\zeta^4 - \zeta^2 + \frac{1}{4(1+e)^2} \le 0$

To get maximum operating range of requencies, $\zeta_{opt} = \sqrt{\frac{1-\sqrt{1-\frac{1}{(1+e)^2}}}{2}}$

Error of seismometer (displacement) :

$$e = \frac{z_{ss\,peak} - y_0}{y_0}$$

$$= r^2 |H(r,\zeta)| - 1$$