1. Charge, Currents, Electric field and magnetic field

E, B Field

It is observed that there is a kind of force between particles which can be described by

- Assigning to every particle a number q, called its charge
- Assigning to every point in space two vector fields called **E** and **B**
  with which the force can then be given by the Lorentz force law:

$$\boxed{\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

Charges densities

- Line charge: The linear charge density λ at a point $\mathbf{r}'$ along the line is defined by
$$\lambda(\mathbf{r}') = \frac{dq}{dl'}$$
  where dq is the amount of charge carried by an infinitesimal segment around $\mathbf{r}'$ with length dl'.

- Surface charge: The surface charge density σ at a point $\mathbf{r}'$ on a plane is defined by
$$\sigma(\mathbf{r}') = \frac{dq}{da'}$$
  where dq is the amount of charge carried by an infinitesimal area element around $\mathbf{r}'$ with area da'.

- Volume charge: The volume charge density ρ at a point $\mathbf{r}'$ inside a volume is defined by
$$\rho(\mathbf{r}') = \frac{dq}{d\tau'}$$
  where dq is the amount of charge carried by an infinitesimal volume element around $\mathbf{r}'$ with volume $d\tau'$.

Current densities

Currents are due to the flow of charges

- Line current: The linear current density (or simply current) **I** at a point $\mathbf{r}'$ along the line is defined a vector with
  Magnitude being the rate of charge flowing through the point, dq/dt

Direction being the same (opposite) as the direction of the motion of the charges if the charges are positive (negative).

If the linear charge density at $\mathbf{r}'$ is λ, then $\boxed{\mathbf{I} = \lambda \mathbf{v}}$, where $\mathbf{v}$ is the velocity of the charges at $\mathbf{r}'$.

- Surface current: The surface current density **K** at a point $\mathbf{r}'$ on a plane is defined as a vector with
  Magnitude being the rate of charge flowing through a line segment perpendicular to the flow and with unit length,
  Direction being the same as the direction of the motion of the charges

  If the surface charge density at $\mathbf{r}'$ is σ, then $\boxed{\mathbf{K} = \sigma \mathbf{v}}$, where $\mathbf{v}$ is the velocity of the charges at $\mathbf{r}'$.

- Volume current: The volume current density **J** at a point $\mathbf{r}'$ inside a volume is defined as a vector with
  Magnitude being the rate of charge flowing through a surface perpendicular to the flow and with unit area,
  Direction being the same as the direction of the motion of the charges

  If the volume charge density at $\mathbf{r}'$ is ρ, then $\boxed{\mathbf{J} = \rho \mathbf{v}}$, where $\mathbf{v}$ is the velocity of the charges at $\mathbf{r}'$.

Conservation of charge and continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \Leftrightarrow \frac{d}{dt}\int_{\mathcal{V}} \rho \, d\tau = -\int_{S} \mathbf{J} \cdot d\mathbf{a}$$

Remark

In this course, we study electromagnetic interactions between charges and currents.

In electromagnetism, the modern interpretation of electromagnetic interaction is that:
1. Charges and currents produce E field and B field
2. Other charges and currents “feel” the E, B field and therefore experience a force described by the Lorentz force law

So there are basically two questions we have to ask:

1. Given E, B field, what is the force experienced by the charges and currents
2. Given charges and currents, what are the E, B fields they produce.

Here we will answer question 1 first.

2. Force experiences by sources inside given fields

2.1 Force experiences by charges inside given E field

By Lorentz force law, given **E**, with no **B**:

- Force on point charge q: $\mathbf{F} = q\mathbf{E}$
- Total force on a set of point charges q_i at $\mathbf{r}_i'$: $\mathbf{F} = \sum_i q_i \mathbf{E}(\mathbf{r}_i')$
- Force on a line P with linear charge density λ: $\mathbf{F} = \int_P \lambda \mathbf{E} dl'$
- Force on a surface S with surface charge density σ: $\mathbf{F} = \int_S \sigma \mathbf{E} da'$.
- Force on a volume $\mathcal{V}$ with volume charge density ρ: $\mathbf{F} = \int_{\mathcal{V}} \rho \mathbf{E} d\tau'$

2.2 Force experiences by currents inside given B field

By Lorentz force law, given **B**, with no **E**:

- Force on a line P with linear current density **I**:

  $$\boxed{\mathbf{F} = \int_P \lambda dl' (\mathbf{v} \times \mathbf{B}) = \int_P \lambda \mathbf{v} \times \mathbf{B} dl' = \int_P \mathbf{I} \times \mathbf{B} dl'}$$

  For a line, **I** must be along the direction of the tangent of the line. Usually, we give the direction of **I** to dl', viz., defining the vector $d\mathbf{l}'$ so that its direction is along the direction of the current: $\mathbf{I}dl' \to Id\mathbf{l}'$

  Hence $\boxed{\mathbf{F} = \int_P (Id\mathbf{l}' \times \mathbf{B})}$

  Usually, I is a constant along the wire → $\boxed{\mathbf{F} = I\int_P (d\mathbf{l}' \times \mathbf{B})}$

- Force on a surface S with surface current density **K**:

  $$\boxed{\mathbf{F} = \int_S \sigma da' (\mathbf{v} \times \mathbf{B}) = \int_S \sigma \mathbf{v} \times \mathbf{B} da' = \int_S \mathbf{K} \times \mathbf{B} da'}.$$

- Force on a volume $\mathcal{V}$ with volume current density **J**:

  $$\boxed{\mathbf{F} = \int_{\mathcal{V}} \rho d\tau' (\mathbf{v} \times \mathbf{B}) = \int_{\mathcal{V}} \rho \mathbf{v} \times \mathbf{B} d\tau' = \int_{\mathcal{V}} \mathbf{J} \times \mathbf{B} d\tau'}$$

3. Fields produced by given sources

3.1 Superposition Principle

It is found that both the E field and B field satisfy the superposition principle. The total field due to different sources is simply the sum of individual fields.

3.2 Coulomb's law

Describes electric fields due to source charges <u>at rest.</u>

- It is found that the E field due to a point charge q is given by

$$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}}$$

- By superposition principle, the field due to a set of discrete charges is

$$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i}$$

For continuous charges distribution:

- Charges along a line with linear charge density $\lambda(\mathbf{r}')$

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_P \frac{\lambda(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}dl'}$$

- Charges on a surface with surface charge density $\sigma(\mathbf{r}')$

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}da'}$$

- Charges filling a volume with volume charge density $\rho(\mathbf{r}')$

$$\boxed{\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\mathscr{r}^2}\hat{\mathscr{r}}d\tau'}$$

The permittivity of free space is $\varepsilon_0 = 8.85\times10^{-12}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}$.

3.3 Biot-Savart Law

Describes magnetic fields due to steady source currents.

There is no "magnetic charge", so there is no Coulomb's law for magnetic field. The sources of magnetic fields are currents

- Line current on a line P: $\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_P \frac{\mathbf{I}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi}\int_P \frac{I d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}}$

  Usually, the current I is constant along the line. Hence

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_P \frac{d\mathbf{l}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}}$$

- Surface current density $\mathbf{K}$ on a surface S:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_S \frac{\mathbf{K}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,da'}$$

- Volume current density $\mathbf{J}$ inside a volume $\mathcal{V}$:

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'}$$

The permeability of free space is $\mu_0 = 4\pi\times 10^{-7}\ \text{N/A}^2$.

Remark

What you learn in phys223 are some new, and more advanced methods to find the fields in some systems.

We need these new methods because direct evaluation from Coulomb's law and Biot-Savart law is either too tedious or, more importantly, may be impossible because the sources are not given explicitly.

For example, in the classic image charge problem with a point charge above an infinite conducting plane, instead of specifying the surface charge density on the plane, it is only given that the plane is grounded. Therefore we cannot find the E field by direct integration of Coulomb's law.

The other methods you learned are all based on solving differential equations.
We shall derive the differential equations satisfied by E, B fields given by the Coulomb's law and the Biot-Savart law.

It can be shown that under very general conditions, a vector field is determined by specifying both its **curl** and **divergence**.

4 Fields, Potentials, Gauss's law and Ampere's law

4.1 Curl and Divergence of Electric Field

From Coulomb's law, one can show that $\boxed{\nabla \times \mathbf{E} = \mathbf{0}}$

The integral form is $\boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = 0}$

One can also derive the Gauss's law:

For a point charge q at $\mathbf{r}'$: $\nabla \cdot \mathbf{E} = \dfrac{q}{\varepsilon_0} \delta^3 (\mathbf{r} - \mathbf{r}')$

For a volume charge density:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}}$$

The integral form of the Gauss's law is

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} \int_{\mathcal{V}} \rho \, d\tau = \frac{Q_{\text{enc}}}{\varepsilon_0}}$$

So basically, you need to solve two equations: $\nabla \times \mathbf{E} = \mathbf{0}$, $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\varepsilon_0}$.

- Method 1: Gauss's Law and Symmetry
  In many systems with symmetry, one can obtain information about the E field by arguments using symmetries, from which the first equation $\nabla \times \mathbf{E} = \mathbf{0}$ is automatically satisfied. Then the field can be obtained very easily by the Gauss's law alone (usually in integral form).
  (For examples, read assignment 2, question 9, 10)

- Method 2: Defining electric potential

  By Stoke's theorem, for any closed path C, $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$.

  This means the line integral of the E field from a point to another is path-independent.

  <u>Electrostatic field is conservative</u>.

  One can therefore define the potential at $\mathbf{r}$ by $\boxed{V(\mathbf{r}) = -\int_{\mathbf{O}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}}$

where O is the reference point at which V is set to 0.

The differential form of the above relation is $\boxed{\mathbf{E} = -\nabla V}$

By defining the electric potential, the first equation $\nabla \times \mathbf{E} = \mathbf{0}$ is automatically satisfied. And the remaining Gauss's law is equivalent to the Poisson's equation

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}.$$

4.2 Curl and Divergence of Magnetic Field

From Biot-Savart law, one can show that $\boxed{\nabla \cdot \mathbf{B} = 0}$

The integral form is $\boxed{\oint_S \mathbf{B} \cdot d\mathbf{a} = 0}$.

One can also derive the Ampere's law $\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}}$

The integral form is $\boxed{\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\text{enc}}}$

So basically, you need to solve two equations: $\nabla \cdot \mathbf{B} = 0$, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.

- Method 1: Ampere's Law and Symmetry
  In many systems with symmetry, one can obtain information about the B field by arguments using symmetries, from which the first equation $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied. Then the field can be obtained very easily by the Ampere's law alone (usually in integral form).
  (For examples, read assignment 7, question 4, 5, 6)

- Method 2: Defining vector potential

  If we construct a vector potential $\mathbf{A}$ so that $\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$

  Then the first equation $\nabla \cdot \mathbf{B} = 0$ is automatically satisfied. And the remaining Ampere's law is equivalent to $\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}$

  By choosing $\mathbf{A}$ so that $\boxed{\nabla \cdot \mathbf{A} = 0}$ (called the Coulomb Gauge), it becomes the Poisson's equation $\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}}$ (Three Poisson's equations in 3D).

5. Solution of Poisson's equation with given boundary conditions

It has been shown that for both E and B field, the second method is reduced to solving the Poisson's equation $\nabla^2\Phi = -\Psi$

5.1 Uniqueness theorems

In this course, we learn two uniqueness theorems only

- First uniqueness theorem:

The solution of the Poisson's equation inside a region $\mathcal{V}$ is unique if the value of Φ on the boundary of $\mathcal{V}$ is specified.

- Second uniqueness theorem:

Inside a region $\mathcal{V}$ with boundaries S_i (number of boundaries can be more than one, which means the region may have "holes" inside), the solution of the Poisson's equation is unique if Φ is constant on all S_i and $\oint_{S_i} \nabla\Phi \cdot d\mathbf{a}$ are all specified.

Therefore, in general, we are to solve the Poisson's equation with particular type of boundary conditions satisfying the above requirements.

Obviously, the above uniqueness theorems also hold for the Laplace's equation $\nabla^2\Phi = 0$, which is just a particular form of Poisson's equation when the source on the right hand side is zero.

The uniqueness theorems implies that if one can somehow construct a solution satisfying the Poisson's equation as well as the specified boundary conditions, then it is the unique solution of the system.

5.2 Solutions of Poisson's equations for simple cases

Constructing the solutions of the Poisson's equation with given boundary condition is in general a difficult task. In this course, we only studied the solutions in a few particularly easy situations:

1. When $\mathcal{V}$ is the entire space with $\Phi = 0$ at infinity

- Solution of $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ for localized ρ, with boundary at infinity on which

  $V = 0$: $$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho(\mathbf{r}')}{\imath} d\tau'}$$

- Solution of $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ for localized $\mathbf{J}$, with boundary at infinity on

  which $\mathbf{A} = \mathbf{0}$: $$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}(\mathbf{r}')}{\imath} d\tau'}$$

It is easy to check that for localized ρ and **J**, the solutions satisfy the required boundary conditions.

To show that they satisfy the Poisson's equation, use $\nabla^2 \frac{1}{\imath} = -4\pi\delta^3(\boldsymbol{\imath})$.

Therefore they are the solutions by first uniqueness theorem.

2. Image charge methods

This method works when the system is simple and is highly symmetric.

- Point charge q at a distance d above a grounded infinite conducting plane

  Using the coordinate system in the lecture notes, the region of interest, $\mathcal{V}$, is

  the space above the xy plane, and the Poisson's equation is

  $\nabla^2 V = -q\delta(\mathbf{r} - d\hat{\mathbf{z}})/\varepsilon_0$ (A delta function peaked at (0, 0, d))

  $\mathcal{V}$ is bounded by the xy plane and infinity, and the boundary condition is

  1) $V = 0$ on the xy plane
  2) $V \rightarrow 0$ at infinity

The solution is the potential due to q and an image charge $-q$ at (0, 0, $-d$):

$$V(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

It is easy to verify that V satisfies both the Poisson's equation and the boundary conditions.

- Point charge q at a distance s outside a grounded conducting sphere with radius R

  The region of interest, $\mathcal{V}$, is the space outside the sphere, and the Poisson's equation is

  $\nabla^2 V = -q\delta(\mathbf{r}-s\hat{\mathbf{z}})/\varepsilon_0$ (A delta function peaked at (0, 0, s))

  $\mathcal{V}$ is bounded by the spherical surface and infinity, and the boundary condition is

  1) $V = 0$ on the spherical surface
  2) $V \rightarrow 0$ at infinity

  The solution is the potential due to q and an image charge $-qR/s$ at (0, 0, R^2/s):

  It is obvious that outside the sphere, Laplacian V is the required delta function at the position of q. The image charge is constructed in a way such that boundary condition 1 is fulfilled, and it is obviously that boundary condition 2 is satisfied by the potential of the two point charges.

In this course, we only consider solutions of the Laplace's equation for more complicated systems

6. Reducing the Poisson's equation into Laplace's equation

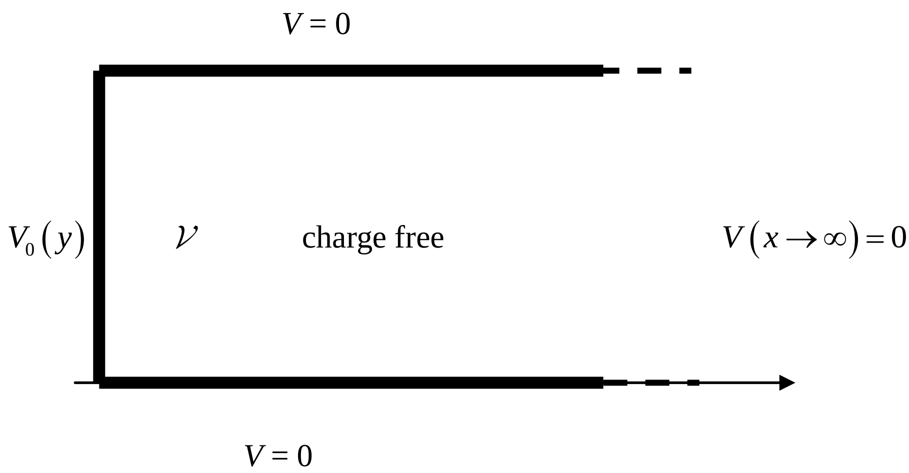
6.1 Electric scalar potential

Since solving the Poisson's equation $\nabla^2 V = -\rho / \varepsilon_0$ with a general ρ and specified boundary conditions in more complicated systems is too difficult for this course. We will only study systems in which the Poisson's equation can be reduced to the simpler Laplace's equation $\nabla^2 V = 0$.

This works when either

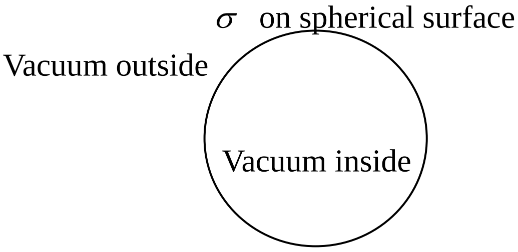
1. The whole region is charge free, so that $\nabla^2 V = 0$ everywhere in $\mathcal{V}$.

   Example 1:



2. The volume charge is confined to some very thin layers, so that $\rho = 0$ everywhere except inside these layers, where it is infinite, so that essentially it can be described by specifying the surface charge densities σ.

   Example 2:



In this case, $\mathcal{V}$ is partitioned into different regions. Inside each region, the Poisson's equation reduces to the Laplace's equation $\nabla^2 V = 0$.

The layers are so thin that we are not interested in the field inside. We will try to march across the layer by "jumping over" it.

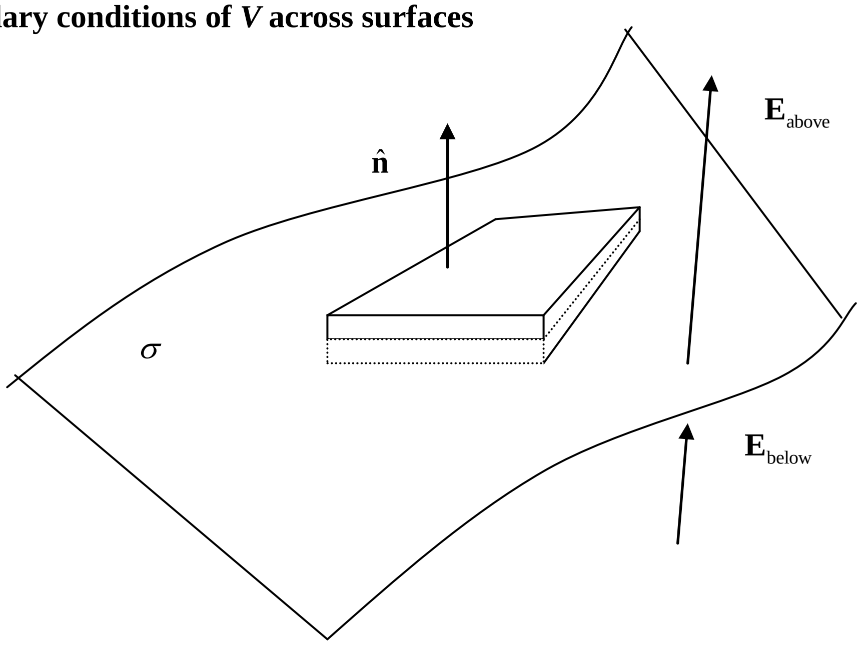
However, although we are not interested in *V* inside the layer, it is not arbitrary because it still needs to satisfy the Poisson's equation with charge density now specified by σ. In other words, *V* on both sides of the layer must be somehow related by σ.

To obtain this relation, recall that the Poisson's equation is equivalent to the two differential equations $\nabla \times \mathbf{E} = 0$, $\nabla \cdot \mathbf{E} = \rho / \varepsilon_0$,

with corresponding integral forms $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$, $\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0}$.

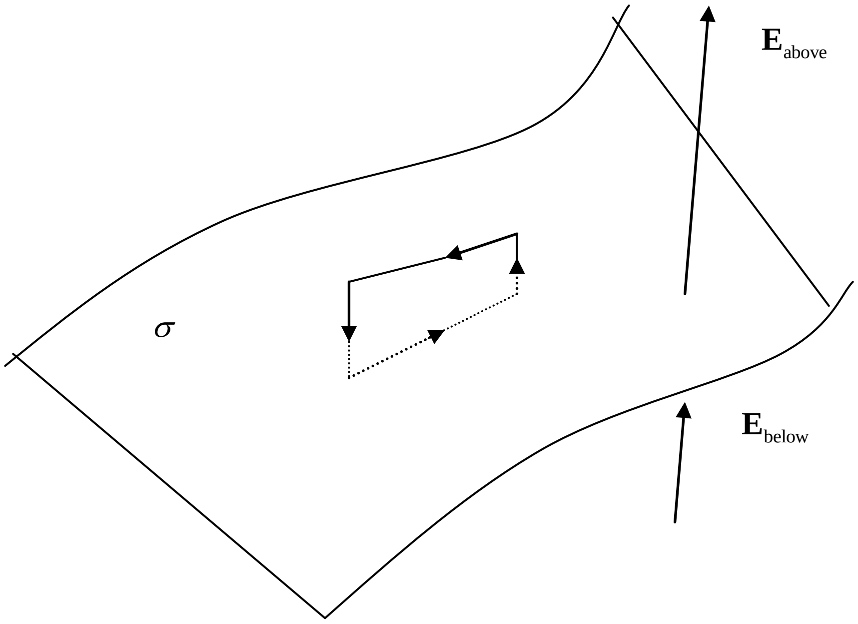
We can go back to the integral form when we "march across" the boundary. This means using the integral forms to <u>match the boundary conditions</u>.

Boundary conditions of *V* across surfaces



$$\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{\text{enc}}}{\varepsilon_0} \Rightarrow \boxed{\left(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\right) \cdot \hat{\mathbf{n}} = \frac{\sigma}{\varepsilon_0}} \Rightarrow \boxed{\frac{\partial V_{\text{above}}}{\partial n} - \frac{\partial V_{\text{below}}}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane and pointing from "below" to "above".



$$\oint_C \mathbf{E}\cdot d\mathbf{l} = 0 \Rightarrow \boxed{\mathbf{E}^{//}_{\text{above}} = \mathbf{E}^{//}_{\text{below}}} \Rightarrow \boxed{V_{\text{above}} = V_{\text{below}}}$$

6.2 Magnetic vector potential

(In this course, we didn't solve **A** using this method. So in 6.2, you only need to understand the boundary condition of the B field across an interface)

The magnetic vector potential can be solved under similar conditions, viz.,

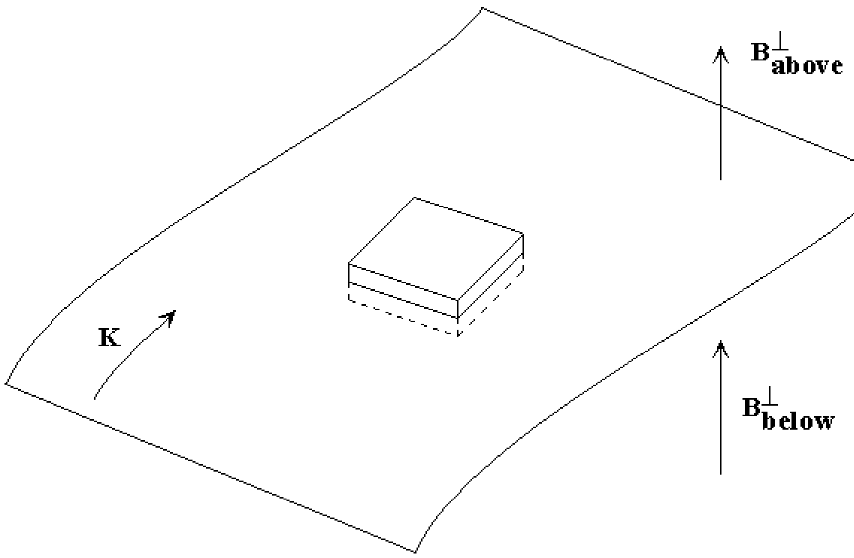
1. when **J** = **0** in the whole region, so that $\nabla^2\mathbf{A} = \mathbf{0}$ everywhere in $\mathcal{V}$, and with **A** specified on the boundaries
2. when the volume current is confined to some very thin layers, so that $\mathbf{J} = \mathbf{0}$ everywhere except inside these layers, where it is infinite, so that essentially it can be described by specifying the surface current densities **K**.

   Similarly, $\mathcal{V}$ is partitioned into different regions. Inside each region, the

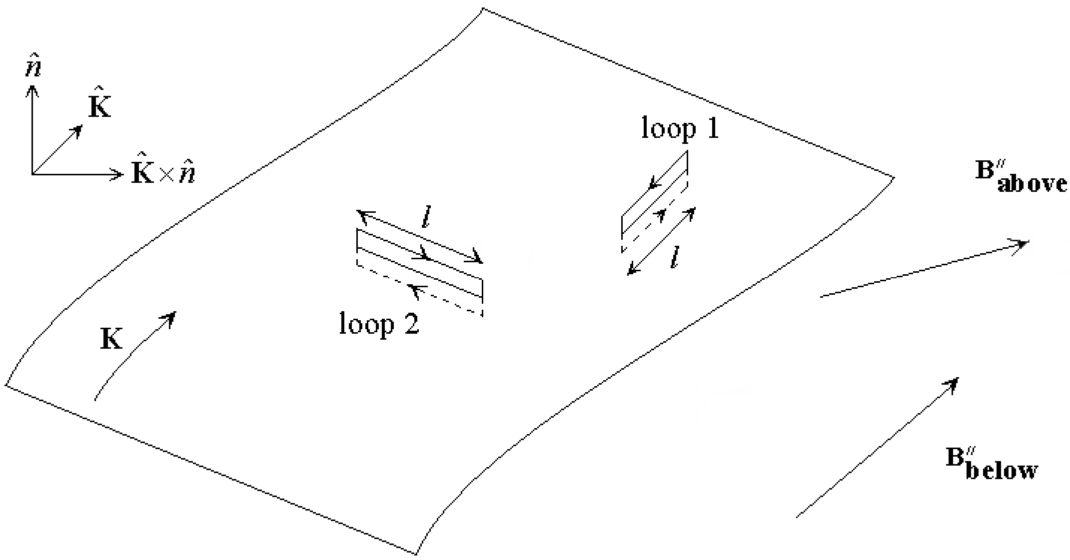
Poisson's equation reduces to the Laplace's equation $\nabla^2\mathbf{A}=\mathbf{0}$, and we should match the boundary condition across the interface, which is now related by **K**.

Boundary conditions across interfaces

In magnetostatics, the two equations we are solving are $\nabla\cdot\mathbf{B}=0$, $\nabla\times\mathbf{B}=\mu_0\mathbf{J}$, with respective integral forms $\oint_S \mathbf{B}\cdot d\mathbf{a}=0$ and $\oint_C \mathbf{B}\cdot d\mathbf{l}=\mu_0 I_{\text{enc}}$



$$\oint_S \mathbf{B}\cdot d\mathbf{a}=0 \Rightarrow \boxed{B^{\perp}_{\text{above}}=B^{\perp}_{\text{below}}}$$



$$\oint_C \mathbf{B}\cdot d\mathbf{l}=\mu_0 I_{\text{enc}} \Rightarrow \boxed{\mathbf{B}^{//}_{\text{above}}-\mathbf{B}^{//}_{\text{below}}=\mu_0\mathbf{K}\times\hat{\mathbf{n}}}$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane and pointing from "below" to “above”.

Using the above boundary condition of the B field, and the fact that $\nabla \cdot \mathbf{A} = 0$, it can be shown that

$$\mathbf{A}^{\|}_{\text{above}} = \mathbf{A}^{\|}_{\text{below}},$$

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.$$

7 Solution of Laplace’s equation (Method of Separation of Variables)

In this course, we will consider only systems whose solution can be obtained by either

1. Solving the Laplace’s equation inside the whole region with specified boundary conditions (example 1 in 6.1), or
2. Solving the Laplace’s equation inside two or more regions and matching the potentials at the interface, as well as the specified boundary conditions at the outermost or innermost boundaries (example 2 in 6.1).

7.1 Spherical coordinates with azimuthal symmetry

General solution:

$$\boxed{V(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)}$$

$P_l(\cos\theta)$ are called the Legendre polynomials, which satisfies the orthogonality relation

$$\boxed{\int_0^{\pi} P_l(\cos\theta) P_m(\cos\theta) \sin\theta d\theta = \frac{2}{2m+1}\delta_{lm}}.$$

The coefficients are chosen to satisfy the boundary conditions.

Boundary conditions:

If $V(a,\theta)$ is specified →

$$\sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta) = V(a,\theta)$$

$$\Rightarrow \int_0^{\pi}\sum_{l=0}^{\infty}\left(A_l a^l + \frac{B_l}{a^{l+1}}\right)P_l(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = \int_0^{\pi}V(a,\theta)P_m(\cos\theta)\sin\theta d\theta$$

$$\Rightarrow \frac{2}{2m+1}\left(A_m a^m + \frac{B_m}{a^{m+1}}\right) = \int_0^{\pi}V(a,\theta)P_m(\cos\theta)\sin\theta d\theta$$

Examples of asymptotic behavior at $r = 0$ or $r \to \infty$

- If V is finite at the centre → $B_l = 0$ for all l.
  (If in particular, $V = 0$ at the center, then $A_0 = 0$)
- If V is zero at infinity → $A_l = 0$ for all l.
- If there is a point charge q in vacuum at the center, then $B_0 = \dfrac{q}{4\pi\varepsilon_0}$ and $B_l = 0$ for $l = 1, 2, 3, \cdots$.
- If $\mathbf{E} \to \mathcal{E}_0\hat{\mathbf{z}}$ when $r \to \infty$, then $A_1 = -\mathcal{E}_0$ and $A_l = 0$ for $l = 2, 3, 4, \cdots$.

Matching across interface:

If there is an interface at $r = a$ on which the surface charge density is $\sigma(\theta)$ →

$$V(a^-,\theta) = V(a^+,\theta)$$

$$\frac{\partial V}{\partial r}(a^+,\theta) - \frac{\partial V}{\partial r}(a^-,\theta) = -\frac{1}{\varepsilon_0}\sigma(\theta)$$

7.2 Cylindrical coordinates with *V* being independent of *z*

General solution

$$V(s,\phi) = A + B\ln s + \sum_{k=1}^{\infty}\left[\left(C_k s^k + D_k s^{-k}\right)\cos k\phi + \left(E_k s^k + F_k s^{-k}\right)\sin k\phi\right]$$

The sine and cosine functions satisfy the orthogonality relations:

$$\int_0^{2\pi}\cos k\phi \sin l\phi \, d\phi = 0 \qquad \text{for all integers } k \geq 0, l > 0.$$

$$\int_0^{2\pi}\cos k\phi \cos l\phi \, d\phi = \begin{cases}\pi\delta_{kl} & \text{for all integers } k, l \text{ when } k \neq 0 \\ 2\pi & \text{when } k = l = 0\end{cases}$$

$$\int_0^{2\pi}\sin k\phi \sin l\phi \, d\phi = \pi\delta_{kl} \qquad \text{for all integers } k, l > 0.$$

The coefficients are chosen to satisfy the boundary conditions.

Boundary conditions:

If $V(a,\phi)$ is specified →

$$A+B\ln a+\sum_{k=1}^{\infty}\left[\left(C_k a^k+D_k a^{-k}\right)\cos k\phi+\left(E_k a^k+F_k a^{-k}\right)\sin k\phi\right]=V\left(a,\phi\right)$$

$$\Rightarrow \int_0^{2\pi}\left\{A+B\ln a+\sum_{k=1}^{\infty}\left[\left(C_k a^k+D_k a^{-k}\right)\cos k\phi+\left(E_k a^k+F_k a^{-k}\right)\sin k\phi\right]\right\}\sin l\phi d\phi=\int_0^{2\pi}V\left(a,\phi\right)\sin l\phi d\phi$$

$$\Rightarrow \pi\left(E_l a^l+F_l a^{-l}\right)=\int_0^{2\pi}V\left(a,\phi\right)\sin l\phi d\phi$$

$$\Rightarrow E_l a^l+F_l a^{-l}=\frac{1}{\pi}\int_0^{2\pi}V\left(a,\phi\right)\sin l\phi d\phi$$

$$\Rightarrow \int_0^{2\pi}\left\{A+B\ln a+\sum_{k=1}^{\infty}\left[\left(C_k a^k+D_k a^{-k}\right)\cos k\phi+\left(E_k a^k+F_k a^{-k}\right)\sin k\phi\right]\right\}\cos l\phi d\phi=\int_0^{2\pi}V\left(a,\phi\right)\cos l\phi d\phi$$

$$\Rightarrow \pi\left(C_l a^l+D_l a^{-l}\right)=\int_0^{2\pi}V\left(a,\phi\right)\cos l\phi d\phi$$

$$\Rightarrow C_l a^l+D_l a^{-l}=\frac{1}{\pi}\int_0^{2\pi}V\left(a,\phi\right)\cos l\phi d\phi \qquad \text{for } l\neq 0$$

$$\Rightarrow \int_0^{2\pi}\left\{A+B\ln a+\sum_{k=1}^{\infty}\left[\left(C_k a^k+D_k a^{-k}\right)\cos k\phi+\left(E_k a^k+F_k a^{-k}\right)\sin k\phi\right]\right\}d\phi=\int_0^{2\pi}V\left(a,\phi\right)d\phi$$

$$\Rightarrow 2\pi\left(A+B\ln a\right)=\int_0^{2\pi}V\left(a,\phi\right)d\phi$$

$$\Rightarrow A+B\ln a=\frac{1}{2\pi}\int_0^{2\pi}V\left(a,\phi\right)d\phi$$

Examples of asymptotic behavior at $s = 0$ or $s\to\infty$

- If the system carries no net charge → $B = 0$.
- If $\mathbf{E}\to\mathcal{E}_0\hat{\mathbf{x}}$ when $s\to\infty$, then all C_l, $E_l = 0$ except $C_1=-\mathcal{E}_0$.

Matching across interface:

If there is an interface at $s = a$ on which the surface charge density is $\sigma(\phi)$ →

$$V\left(a^-,\phi\right)=V\left(a^+,\phi\right)$$

$$\frac{\partial V}{\partial s}\left(a^+,\phi\right)-\frac{\partial V}{\partial s}\left(a^-,\phi\right)=-\frac{1}{\varepsilon_0}\sigma\left(\phi\right)$$

7.3 Cartesian coordinates

The situation is more complicated in Cartesian coordinates because the form of solutions depends explicitly on the boundary. For this part, please refer to lecture note chap3_2 for details.

8 Multipole Expansion

It is based on the following expansion of the potential

$$\boxed{\frac{1}{\mathscr{r}} = \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \sum_{n=0}^{\infty} \frac{r'^n}{r^{n+1}} P_n\left(\cos\theta'\right) = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n\left(\cos\theta'\right)}$$

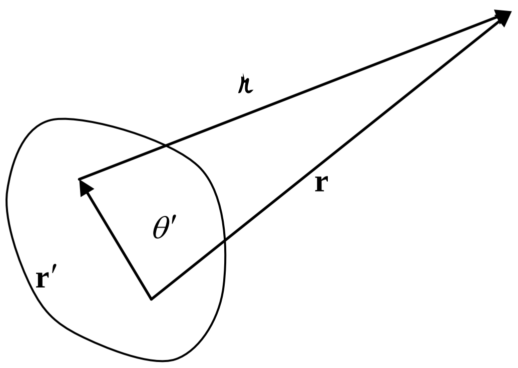
For a localized source distribution, we can use the above formula to expand the potential when $r > r'$.

The series obtained gives good approximation by keeping leading order terms when we observe the potential at a very large distance compared with the size of the source, viz., $r \gg r'$.

8.1 Electric potential

Consider a localized charge distribution given by $\rho\left(\mathbf{r}'\right)$. The potential at $\mathbf{r}$ is

$$V\left(\mathbf{r}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho\left(\mathbf{r}'\right)d\tau'$$



Using $\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n\left(\cos\theta'\right)$, the potential at $\mathbf{r}$ is therefore

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int r'^n P_n(\cos\theta') \rho(\mathbf{r}') d\tau'$$

- Monopole Term $n = 0$

$$\begin{aligned} V_{\text{mon}}(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \frac{1}{r} \int \rho(\mathbf{r}') d\tau' \\ &= \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} \end{aligned}$$

where $Q = \int \rho(\mathbf{r}') d\tau'$ is the total charge or the monopole moment.

This is the dominant term at large distance if $Q \neq 0$. Under this approximation, all the charges are considered to be located at the origin.

The monopole field is $\mathbf{E}_{\text{mon}}(\mathbf{r}) = -\nabla V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}$.

Notice that the monopole moment Q is obviously independent of the point we choose as the origin.

- Dipole Term $n = 1$

$$\begin{aligned} V_{\text{dip}}(\mathbf{r}) &= \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int r' \cos\theta' \rho(\mathbf{r}') d\tau' \\ &= \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int \hat{\mathbf{r}} \cdot \mathbf{r}' \rho(\mathbf{r}') d\tau' \\ &= \frac{1}{4\pi\varepsilon_0} \frac{\hat{\mathbf{r}}}{r^2} \cdot \int \mathbf{r}' \rho(\mathbf{r}') d\tau' \end{aligned}$$

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}$$

where $\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$

is the dipole moment of the distribution. The dipole moment so defined is a vector, which depends only on the location, size, shape and distribution of the sources, but independent of the observation point.

For n point charges $q_1, q_2, \cdots, q_n$ at $\mathbf{r}'_1, \mathbf{r}'_2, \cdots, \mathbf{r}'_n$, respectively,

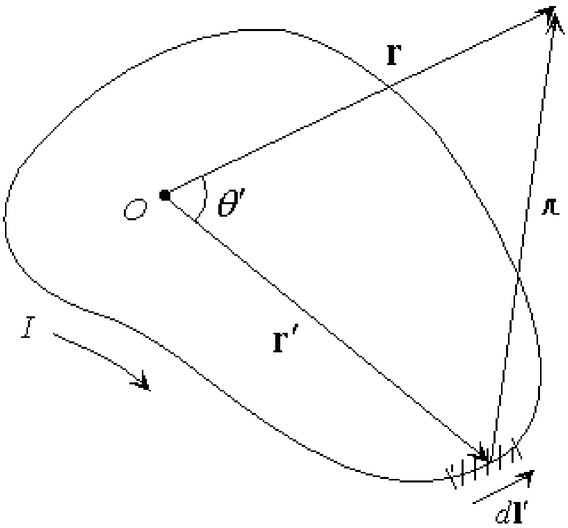
$$\mathbf{p} = \sum_{i=1}^{n} q_i \mathbf{r}'_i$$

Notice that the dipole moment depends *also on the choice of origin*.
However, if Q = 0, then the dipole moment becomes independent of the choice of origin.
The dipole term becomes the dominant term if Q = 0 while $\mathbf{p} \neq \mathbf{0}$.

The electric dipole field is $$\boxed{\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}\right]}.$$

8.2 Magnetic vector potential

For a linear current flowing in a localized wire with uniform current I.



$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}}{\mathscr{r}}dl' = \frac{\mu_0 I}{4\pi}\int \frac{1}{\mathscr{r}}d\mathbf{l}'$$
$$= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\,d\mathbf{l}'$$

Monopole: n = 0

$$\mathbf{A}_{\text{mon}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r}\oint d\mathbf{l}'$$

For a closed loop $\oint d\mathbf{l}' = \mathbf{0}$

$$\therefore \quad \boxed{\mathbf{A}_{\text{mon}}(\mathbf{r}) = \mathbf{0}}$$

Dipole: n = 1

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\frac{1}{r^2}\oint r' \cos\theta \, d\mathbf{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2}\oint (\hat{\mathbf{r}} \cdot \mathbf{r}') \, d\mathbf{l}'$$

$$= \frac{\mu_0}{4\pi r^2}\left(I\int d\mathbf{a}'\right)\times\hat{\mathbf{r}}$$

$$\boxed{\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}}$$

where the magnetic dipole moment is defined by $\boxed{\mathbf{m} = I\mathbf{a} = I\int d\mathbf{a}'}$

The dipole field is $\boxed{\mathbf{B}_{\text{dip}} = \nabla\times\mathbf{A}_{\text{dip}} = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]}$.

9 EM field inside matter

Inside matter, there are a lot of tiny objects in motions (atoms, molecules, electrons, ions, nucleons, etc.), giving rise to charge distributions and current distributions. However, the size of these objects are all very small when we observe the field at a macroscopic distance, so that it is feasible to keep only the leading order term in the multipole expansion.

For current distributions, we know that there is no monopole moment. Hence the leading order term is the dipole term.
For charge distributions, since at a macroscopic scale, matter carries no net charge. Hence the leading order term is also the dipole term.

In conclusion, ordinary matter can be consider as consisting of a large number of tiny electric and/or magnetic dipoles.

Polarization $\boxed{\mathbf{P} = \text{ electric dipole moment per unit volume}}$

Magnetization $\boxed{\mathbf{M} = \text{ magnetic dipole moment per unit volume}}$

9.1 Fields due to P and M

Knowing the potential due to ideal point dipoles, one can easily obtain the potential due to an object with **P** and/or **M** by considering every small volume element as ideal point dipole and summing the potentials of all these dipoles:

For **P**:
$$\boxed{V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma_b}{\mathscr{r}} da' + \frac{1}{4\pi\varepsilon_0}\int_{\mathcal{V}} \frac{\rho_b}{\mathscr{r}} d\tau'}$$

where the surface bound charge density is $\boxed{\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}}$

and the volume bound charge density is $\boxed{\rho_b = -\nabla\cdot\mathbf{P}}$

For **M**:
$$\boxed{\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\oint_S \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}} da' + \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}} d\tau'}$$

where the surface bound current density $\boxed{\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}}$

and the volume bound current density $\boxed{\mathbf{J}_b = \nabla \times \mathbf{M}}$

(For physical interpretation, refer to lecture notes)

Gauss's law in matter with polarization: $\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\left(\rho_f + \rho_b\right)}$

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f \Leftrightarrow \oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f\text{enc}}}$$

where the electric displacement is defined by $\boxed{\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}}$

Ampere's law in matter with magnetization: $\boxed{\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J}_f + \mathbf{J}_b\right)}$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}_f \Leftrightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \mathbf{I}_{f\text{enc}}}$$

where the H-field is defined by $\boxed{\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}}$

Note that $\boxed{\nabla \times \mathbf{D} = \nabla \times \mathbf{P}} \neq \mathbf{0}$, $\boxed{\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}} \neq 0$.

With new unknown fields **P** and **M**, we can't solve E and B unless additional information about the relation between **P** and **E**, and **M** and **B** are provided. These are called the constitutive relations: **P**(**E**), **M**(**B**).

Differential Equations for solving D:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_f \Leftrightarrow \oint_S \mathbf{D} \cdot d\mathbf{a} = Q_{f\text{enc}}},$$

$$\boxed{\nabla \times \mathbf{D} = \nabla \times \mathbf{P} \Leftrightarrow \oint_C \mathbf{D} \cdot d\mathbf{l} = \oint_C \mathbf{P} \cdot d\mathbf{l}}$$

If the system has symmetries so that $\nabla \times \mathbf{D} = \nabla \times \mathbf{P}$ is automatically taken care of, then one can use the Gauss's law of **D** (usually in integral form) to obtain the D field easily. (Examples in lecture notes 4.2)

Differential Equations for solving H:

$$\boxed{\nabla\times\mathbf{H}=\mathbf{J}_f \Leftrightarrow \oint_C \mathbf{H}\cdot d\mathbf{l}=\mathbf{I}_{f\text{enc}}},$$

$$\boxed{\nabla\cdot\mathbf{H}=-\nabla\cdot\mathbf{M} \Leftrightarrow \oint_S \mathbf{H}\cdot d\mathbf{a}=-\oint_S \mathbf{M}\cdot d\mathbf{a}}$$

If the system has symmetries so that $\nabla\cdot\mathbf{H}=-\nabla\cdot\mathbf{M}$ is automatically taken care of, then one can use the Ampere's law of **H** (usually in integral form) to obtain the H field easily. (Example in lecture notes 6.2)

Boundary conditions across interfaces

In the general case, you have to solve the above differential equations directly, which is usually difficult. The general method is not taught in this course. Again, we will only study systems in which the right hand side of the equations, viz.,

$\rho_f, \mathbf{J}_f, \nabla\times\mathbf{P}, \nabla\cdot\mathbf{M}$ are zero everywhere except on some interfaces.

In this case, one will use the integral form of the differential equations to obtain the relation of the fields on the two sides of an interface:

- Across an interface of two different media with free charge density σ_f, the integral forms yield the following boundary conditions:

$$\boxed{D^{\perp}_{\text{above}}-D^{\perp}_{\text{below}}=\sigma_f},\quad \boxed{\mathbf{D}^{//}_{\text{above}}-\mathbf{D}^{//}_{\text{below}}=\mathbf{P}^{//}_{\text{above}}-\mathbf{P}^{//}_{\text{below}}}$$

- Across an interface of two different media with free current density $\mathbf{K}_f$, the integral forms yield the following boundary conditions:

$$\boxed{\mathbf{H}^{//}_{\text{above}}-\mathbf{H}^{//}_{\text{below}}=\mathbf{K}_f\times\hat{\mathbf{n}}},\quad \boxed{H^{\perp}_{\text{above}}-H^{\perp}_{\text{below}}=-\left(M^{\perp}_{\text{above}}-M^{\perp}_{\text{below}}\right)}$$

9.2 Simple Constitutive Relations

The relations between fields and **P** or **M** depend on how the dipoles respond to the applied fields. Inside a lot of materials, the constitutive relations turn out to be simple. We studied two particularly simple cases

1. "Hard" materials with uniform "frozen-in" polarization or magnetization:

In this kind of materials, P and/or M are assumed to be permanent, not affected by the applied field when the field is weak. A common example is a magnet, with "frozen-in" magnetization, or a less common one, its electric counterpart, the electret.

2. Linear materials with $\mathbf{P} \propto \mathbf{E}$ and $\mathbf{M} \propto \mathbf{B}$.

Before I explain why the above two cases are easy to solve, let's first investigate why the second kind of materials is common.

9.3 Effect of external fields on matter

Effect 1:

It can be shown that

- Inside external E field **E**, an ideal electric point dipole **p** will experience a force

$$\boxed{\mathbf{F} = (\mathbf{p} \cdot \nabla)\mathbf{E}}$$

and a torque $$\boxed{\mathbf{N} = \mathbf{p} \times \mathbf{E}}$$

- Inside external B field **B**, an ideal magnetic point dipole **m** will experience a force

$$\boxed{\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})}$$

and a torque $$\boxed{\mathbf{N} = \mathbf{m} \times \mathbf{B}}$$

Usually the dipoles are so small that the fields acting on them are essentially uniform. Hence the forces, which involve the spatial derivatives of the fields, are zero.

But the torques are non-zero even in uniform fields. They try to align the dipoles in the directions of the fields.

Without the applied fields, the dipoles inside matter have random directions, and there is no net macroscopic dipole moment.

With the applied field, there will more a net amount of dipole pointing at the same direction as the field.

Effect 2:

Another effect is that the dipoles are not absolutely rigid bodies, nor are they actually points. They have internal structures, which will be altered by the fields.

For atoms with no dipole moment initially, an applied electric field will push nucleus and pull the electron clouds, inducing a dipole in the same direction as the field.

For magnetic dipole moment due to orbital motion of the electrons, without the applied B field, they will have random directions. An applied magnetic field will not only try to align them, but also alter the orbits of the electrons and increase the magnetic dipole moment in the opposite direction of the field.

9.4 Linear Materials

In conclusion, these effects lead to induced dipole moments either along the same direction of the field or in the opposite direction.

When the applied fields are weak, to first order approximation, the dipole moments will have a linear relation to the field:

$$\mathbf{P} \propto \mathbf{E} \quad \text{and} \quad \mathbf{M} \propto \mathbf{B}$$

By definition of D and H, this implies

$\mathbf{P} \propto \mathbf{E} \propto \mathbf{D}$ and $\mathbf{M} \propto \mathbf{B} \propto \mathbf{H}$

Define $\boxed{\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}}$ χ_e :electric susceptibility

Materials obeying this relation are called linear dielectrics.

Hence $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (1 + \chi_e) \mathbf{E}$

Define $\boxed{\varepsilon = \varepsilon_0 (1 + \chi_e)}$ so that $\boxed{\mathbf{D} = \varepsilon \mathbf{E}}$ ε : Permittivity of the material

For linear magnetic materials,

Define $\boxed{\mathbf{M} = \chi_m \mathbf{H}}$ χ_m :magnetic susceptibility

Hence $\mathbf{B} = \mu_0 \left(\mathbf{H} + \mathbf{M} \right) = \mu_0 \left(1 + \chi_m \right) \mathbf{H}$

Define $\boxed{\mu = \mu_0 (1 + \chi_m)}$ so that $\boxed{\mathbf{H} = \dfrac{\mathbf{B}}{\mu}}$ μ :permeability of the material

10 Solving Laplace's equation in system with Polarization and Magnetization

The simple systems involving **P** and/or **M** we solved in this course are all of this form:

- A region which is divided into several parts, separated by interfaces.
- Inside each region, we have either linear materials or uniform "frozen-in" **P** or **M**, and
- with no free sources.

Here is the general technique of solving this kind of systems:

10.1 Polarization

When there is no free charges, $\nabla\cdot\mathbf{D}=0$ (no free charges)
Besides, when **P** is uniform, $\nabla\cdot\mathbf{P}=0$.

Hence, $\nabla\cdot\mathbf{E}=\nabla\cdot\left(\dfrac{\mathbf{D}-\mathbf{P}}{\varepsilon_0}\right)=0$

On the other hand, when the material is linear, $\mathbf{E}\propto\mathbf{D}\Rightarrow\nabla\cdot\mathbf{E}\propto\nabla\cdot\mathbf{D}=0$.
In both cases, $\nabla\cdot\mathbf{E}=0$. Together with $\nabla\times\mathbf{E}=0$, the problem reduces to solving the Laplace's equation in different regions.

The boundary conditions one has to match:

The BC $\mathbf{D}^{//}_{\text{above}}-\mathbf{D}^{//}_{\text{below}}=\mathbf{P}^{//}_{\text{above}}-\mathbf{P}^{//}_{\text{below}}$

is equivalent to $\mathbf{D}^{//}_{\text{above}}-\mathbf{P}^{//}_{\text{above}}=\mathbf{D}^{//}_{\text{below}}-\mathbf{P}^{//}_{\text{below}}\Leftrightarrow\mathbf{E}^{//}_{\text{above}}=\mathbf{E}^{//}_{\text{below}}$

which is equivalent to the continuity of *V*: $\boxed{V_{\text{above}}=V_{\text{below}}}$.

However, the other BC is more difficult: $D^{\perp}{}_{\text{above}}-D^{\perp}{}_{\text{below}}=0$.

We need to rewrite this in terms of **E** and **P**.

If the medium is linear with permittivity ε, $\boxed{D^{\perp}=\varepsilon E^{\perp}=-\varepsilon\dfrac{\partial V}{\partial n}}$.

If the medium has uniform polarization **P**, $\boxed{D^{\perp} = \varepsilon_0 E^{\perp} + P^{\perp} = -\varepsilon_0 \frac{\partial V}{\partial n} + P^{\perp}}$

10.2 Magnetization

When there is no free currents, $\nabla \times \mathbf{H} = \mathbf{0}$ (no free currents).
Hence, one can always define a magnetic scalar potential of **H**, *W*, so that

$$\boxed{\mathbf{H} = -\nabla W}$$

Because $\nabla \cdot \mathbf{H} = -\nabla \cdot \mathbf{M}$, then

$$\nabla^2 W = \nabla \cdot \mathbf{M}.$$

In other words, *W* obeys the Poisson's equation, with $-\nabla \cdot \mathbf{M}$ as the "source".
Inside a region where **M** is uniform, $\nabla \cdot \mathbf{M} = 0$.
On the other hand, if the medium is linear, then $\mathbf{M} \propto \mathbf{B} \Rightarrow \nabla \cdot \mathbf{M} \propto \nabla \cdot \mathbf{B} = 0$.
In both cases, *W* satisfies the Laplace's equation $\nabla^2 W = 0$.

Again, we need to derive the new boundary conditions to match:

With no free currents, the first boundary condition $\mathbf{H}^{//}_{\text{above}} - \mathbf{H}^{//}_{\text{below}} = \mathbf{K}_f \times \hat{\mathbf{n}} = \mathbf{0}$ is

equivalent to the continuity of *W* across an interface $\boxed{W_{\text{above}} = W_{\text{below}}}$

The second boundary condition is $H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -\left(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}}\right)$, which can be

rewritten as $H^{\perp}_{\text{above}} + M^{\perp}_{\text{above}} = H^{\perp}_{\text{below}} + M^{\perp}_{\text{below}}$.

Inside a medium with uniform magnetization, $\boxed{H^{\perp} + M^{\perp} = -\frac{\partial W}{\partial n} + M^{\perp}}$.

Inside a linear medium with permeability μ, $\mathbf{M} = \chi_m \mathbf{H} = \left(\frac{\mu}{\mu_0} - 1\right)\mathbf{H}$, hence

$$\boxed{H^{\perp} + M^{\perp} = \frac{\mu}{\mu_0} H^{\perp} = -\frac{\mu}{\mu_0}\frac{\partial W}{\partial n}}.$$

(Read assignment 8, questions 6, 7, 8, 9 for examples of using *W* to solve obtain **B**)