Unitary Transformations in $L_2(I, \mathbb{C})$ - Bochner's Theorem

Let $I = [a, b]$, $-\infty \leq a \leq 0 \leq b \leq \infty$ and let L_2 represent $L_2(I, \mathbb{C})$.

Theorem (A): Unitary transformations (operators) on L_2 can be represented by integrations with a certain kernal function

(A) Let $\mathcal{U} : L_2 \to L_2$ be a unitary transformation. Then there are associated two functions $K(\xi, x)$ and $H(\xi, x)$, each defined $I \times I \to \mathbb{C}$, with $K(\xi, \cdot)$ and $H(\xi, \cdot) \in L_2$ such that for each $f \in L_2$ and $g = \mathcal{U}f$,

$$(1) \qquad \int_0^{\xi} \mathrm{g(x)dx} = \int_a^b \overline{\mathrm{K}(\xi, \mathrm{x})}\mathrm{f(x)dx} \quad \text{and} \quad \int_0^{\xi} \mathrm{f(x)dx} = \int_a^b \overline{\mathrm{H}(\xi, \mathrm{x})}\mathrm{f(x)dx}.$$

These functions further satisfy

$$\left.\begin{array}{ll} (2) & \int_a^b \overline{K(\xi,x)}K(\eta,x)dx \\ (3) & \int_a^b \overline{H(\xi,x)}H(\eta,x)dx \end{array}\right\} = \begin{cases} \min\{|\xi|, |\eta|\}, & \xi\eta \geq 0 \\ 0, & \xi\eta < 0 \end{cases},$$

and

$$(4) \qquad \int_0^{\eta} \mathrm{K}(\xi, \mathrm{x})\mathrm{dx} = \int_0^{\xi} \overline{\mathrm{H}(\eta, \mathrm{x})}\mathrm{dx}.$$

(B) Conversely, every pair of such H and K satisfying (2) through (4) generate, through (1) a unitary transformation and it's inverse. Every pair of kernal fns, satisfying some restrictions, corresponds to a unitary transformation (operator) on L_2

Proof: **(A)** For $\xi, x \in (a, b)$, define the step function

$$e_{\xi}(x) = \begin{cases} 1 & 0 \leq x \leq \xi, \\ -1 & \xi \leq x \leq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Also define

$$H(\xi, x) = \mathcal{U}e_{\xi}(x) \quad \text{and} \quad \mathrm{K}(\xi, \mathrm{x}) = \mathcal{U}^{-1}\mathrm{e}_{\xi}(\mathrm{x}). \tag{5}$$

Let $f \in \mathcal{H}$ and $g = \mathcal{U}f$. Then for the inner product in L_2, $(*, \cdot)$, we have the following,

$$\begin{aligned} (g, e_{\xi}) &= (\mathcal{U}f, e_{\xi}) = (f, \mathcal{U}^{-1}e_{\xi}), &(6)\\ (f, e_{\xi}) &= (\mathcal{U}^{-1}g, e_{\xi}) = (g, \mathcal{U}e_{\xi}). &(7) \end{aligned}$$

Note:

$$(g, e_{\xi}) = \int_I g(x)e_{\xi}(x)dx = \begin{cases} -\int_{\xi}^0 g(x)dx, & \xi < 0, \\ \int_0^{\xi} g(x)dx, & \xi \geq 0, \end{cases} = \int_0^{\xi} g(x)dx,$$

in each case.

Thus, from (6) and (7) we obtain, respectively,

$$\begin{aligned} \int_0^{\xi} g(x)dx &= \int_a^b \overline{K(\xi, x)}f(x)dx, &(8)\\ \int_0^{\xi} f(x)dx &= \int_a^b \overline{H(\xi, x)}g(x)dx. &(9) \end{aligned}$$

For the special choices,

$$f = \mathcal{U}^{-1}e_{\eta} \quad \text{and} \quad \mathrm{g} = \mathcal{U}\mathrm{f} = \mathrm{e}_{\eta},$$

(8) we obtain

$$\int_a^b \overline{K(\xi,x)}K(\eta,x)dx = \int_0^\xi e_\eta = \begin{cases} \min(\xi,\eta), & \xi \geq 0,\ \eta \geq 0, \\ \min(|\xi|,|\eta|), & \xi < 0,\ \eta < 0, \\ 0 & \xi\eta < 0 \end{cases}$$

$$= \begin{cases} \min(|\xi|,|\eta|), & \xi\eta \geq 0, \\ 0 & \xi\eta < 0 \end{cases}. \tag{10}$$

This proves part (2). Choosing

$$f = \mathcal{U}g = e_\eta \quad \text{and} \quad \mathrm{g} = \mathcal{U}\mathrm{f} = \mathcal{U}e_\eta,$$

and using (9) one obtain, by the same computations, part (3). To prove (4) note that $\mathcal{U}^{-1} = \mathcal{U}^*$ so that the following holds,

$$L \doteq (\mathcal{U}^{-1}e_\xi, e_\eta) = (e_\xi, \mathcal{U}e_\eta) \doteq R$$

From (5),

$$L = (K(\xi,\cdot), e_\xi(\cdot)) = \int_a^b K(\xi,x)e_\xi(x)dx = \int_0^\eta K(\xi,x)dx$$

$$R = (e_\xi(\cdot), H(\eta,\cdot)) = \int_a^b e_\xi(x)\overline{H(\eta,x)}dx = \int_0^\xi \overline{H(\xi,x)}dx,$$

and therefore

$$\int_0^\eta K(\xi,x)dx = \int_0^\xi \overline{H(\xi,x)}dx, \tag{11}$$

and this completes the proof of part (A).
(B) For the converse, assume we have functions K and H satisfying (2), (3) and (4). Define two transformations $\mathcal{U}$ and $\mathcal{V}$ by

$$\mathcal{U}e_\xi(x) = H(\xi,x) \qquad \text{and} \qquad \mathcal{V}e_\xi(\mathrm{x}) = \mathrm{K}(\xi,\mathrm{x}).$$

Then

$$\begin{aligned} (2) &\implies (\mathcal{V}e_\xi, \mathcal{V}e_\eta) = (e_\xi, e_\eta) \\ (3) &\implies (\mathcal{U}e_\xi, \mathcal{U}e_\eta) = (e_\xi, e_\eta) \\ (4) &\implies (\mathcal{V}e_\xi, e_\eta) = (e_\xi, \mathcal{U}e_\eta). \end{aligned}$$

Next let u and v be step functions. Then

$$u(x) = \sum_{k=1}^N \alpha_k e_{\xi_k}(x - \beta_k),$$

and similarly for v. Then by the b–linearity of all the inner products involved, $\mathcal{U}$ and $\mathcal{V}$ can be extended to all step functions in L_2 and

$$(\mathcal{V}u, \mathcal{V}v) = (u,v), \quad (\mathcal{U}u, \mathcal{U}v) = (u,v) \quad \text{and} \quad (\mathcal{V}\mathrm{u}, \mathrm{v}) = (\mathrm{u}, \mathcal{U}\mathrm{v}). \tag{12}$$

But step functions are dense in L_2 so $\mathcal{U}$ and $\mathcal{V}$ can be extended by continuity to L_2 so that (12) holds for L_2 functions u and v.

Therefore from (12) we see

$$\mathcal{V}^*\mathcal{V} = I = \mathcal{U}^*\mathcal{U} \qquad \text{and} \qquad \mathcal{U} = \mathcal{V}^*.$$

Thus $\mathcal{V}^* = \mathcal{U}^{-1}$ and $\mathcal{U} = \mathcal{V}^*$ is an isometry.