III (Point)

Methods - Estimation (Methods of Evaluating Estimators)

1° luck (Cor)
2° solve (Cor)
3° conditional (L-S)
4° Information inequality
5° Linear dependence

Def: $\underline{X} \sim p(\underline{x};\theta)$, $\theta \in \Theta$, statistic $T(\underline{X})$ is <u>unbiased</u> for $q(\theta)$ if $\mathbb{E}_\theta T(\underline{X}) = q(\theta)$, $\forall \theta \in \Theta$

Thm: Given $MSE_\theta T(\underline{X}) \equiv \mathbb{E}_\theta (T(\underline{X}) - q(\theta))^2$, then

$$MSE_\theta(T(\underline{X})) = Var_\theta T(\underline{X}) + Bias_\theta^2 T(\underline{X})$$

where $Bias_\theta T(\underline{X}) \equiv \mathbb{E}_\theta T(\underline{X}) - q(\theta)$.

Def: Estimator $S(\underline{X})$ is <u>uniformly minimum variance, unbiased</u> for $q(\theta)$, (UMVU)
if $\mathbb{E}_\theta S(X) = q(\theta)$ and

$$Var_\theta S(\underline{X}) \le Var_\theta U(\underline{X}), \ \forall \text{ unbiased } U(\underline{X}) \text{ of } q(\theta), \ \forall \theta \in \Theta.$$

Notes: 1° May not exist

$X \sim$ Binomial(n, θ), there's no unbiased estimator for $q(\theta) = \frac{\theta}{1-\theta}$. This place supposed $T(\underline{X}) = g(\underline{X})$

$\mathbb{E}_\theta\, g(\underline{X}) = \ ?$

2° May not well behaved

$X \sim$ Poisson(θ), the UMVU estimator is $g(X) = (-1)^X$

Lemma: r.v. X satisfies $\mathbb{E}X^2 < \infty$, then

$$Var(\mathbb{E}(X|Y)) \le Var X \quad , \forall \text{ r.v. } Y.$$

proof: $Var X = \mathbb{E}\{Var\{X|Y\}\} + Var(\mathbb{E}(X|Y)) \ge Var(\mathbb{E}(X|Y))$. $\square$

Thm: (Rao-Blackwell Theorem)

If $T(\underline{X})$ is sufficient for θ, and $S(\underline{X})$ is an estimator for $q(\theta)$, with $\mathbb{E}_\theta |S(\underline{X})| < \infty$, $\forall \theta \in \Theta$. Let $S^*(\underline{X}) \equiv \mathbb{E}(S(\underline{X}) | T(\underline{X}))$, then

$$\mathbb{E}_\theta (S^*(\underline{X}) - q(\theta))^2 \le \mathbb{E}_\theta (S(\underline{X}) - q(\theta))^2$$

proof: $\text{Bias}_\theta S^*(\underline{X}) = \mathbb{E}_\theta S^*(\underline{X}) - q(\theta)$

$$= \mathbb{E}_\theta(\mathbb{E}(S(\underline{X}) \mid T(\underline{X}))) - q(\theta)$$

$$= \mathbb{E}_\theta S(\underline{X}) - q(\theta)$$

$$= \text{Bias}_\theta S(\underline{X})$$

$$\text{Var}\, S^*(\underline{X}) = \text{Var}_\theta(\mathbb{E}(S(\underline{X}) \mid T(\underline{X}))) \le \text{Var}_\theta S(\underline{X})$$

$\therefore \text{MSE}_\theta S^*(\underline{X}) \le \text{MSE}_\theta S(\underline{X})$ $\square$.

Note: $X_1, \ldots, X_n$ iid $N(\mu, 1)$

$S(\underline{X}) = X_1$ is ~~sufficient~~ unbiased for μ.

$T(\underline{X}) = \sum_{i=1}^n X_i$ is sufficient for μ.

Then $S^*(\underline{X}) = E(X_1 \mid \sum_{i=1}^n X_i)$

$$= \frac{1}{n} E(\sum_{i=1}^n X_i \mid \sum_{i=1}^n X_i)$$

$$= \frac{1}{n} \sum_{i=1}^n X_i$$

$$= \bar{X}$$

Thm: (Lehmann-Scheffe Theorem)

Let $T(\underline{X})$ be a complete, sufficient statistic for θ, let $S(\underline{X})$ be unbiased for $q(\theta)$, then $S^*(\underline{X}) = \mathbb{E}(S(\underline{X}) \mid T(\underline{X}))$ is UMVU for $q(\theta)$

proof: $\mathbb{E}_\theta S^*(\underline{X}) = \mathbb{E}_\theta S(\underline{X}) = q(\theta)$

$\therefore S^*(\underline{X})$ is unbiased for $q(\theta)$.

$\forall$ unbiased estimator $U(\underline{X})$ for $q(\theta)$,

let $g_1(T(\underline{X})) \equiv \mathbb{E}(U(\underline{X}) \mid T(\underline{X}))$

$$g_2(T(\underline{X})) \equiv \mathbb{E}(S(\underline{X}) \mid T(\underline{X}))$$

$$g(T(\underline{X})) = g_1(T(\underline{X})) - g_2(T(\underline{X}))$$

$$
\begin{aligned}
\mathbb{E}_\theta\, g(T(\underline{X})) &= \mathbb{E}_\theta\, g_1(T(\underline{X})) - \mathbb{E}_\theta\, g_2(T(\underline{X})) \\
&= \mathbb{E}_\theta\, U(\underline{X}) - \mathbb{E}_\theta\, S(\underline{X}) \\
&= q(\theta) - q(\theta) \\
&= 0 \qquad \forall\, \theta \in \Theta
\end{aligned}
$$

$\because$ $T(\underline{X})$ is complete

$\therefore$ $g(t) = 0$, i.e. $\mathbb{E}(S(\underline{X}) \mid T(\underline{X})) = \mathbb{E}(U(\underline{X}) \mid T(\underline{X}))$

$\therefore$ $Var_\theta S^*(\underline{X}) = Var_\theta \mathbb{E}(S(\underline{X}) \mid T(\underline{X})) = Var_\theta \mathbb{E}(U(\underline{X}) \mid T(\underline{X})) \leq Var_\theta U(\underline{X})$

$\therefore$ $S^*(\underline{X})$ is UMVU for $q(\theta)$. $\square$.

Cor: If $S(X)$ is a function of a complete, sufficient statistic, it is UMVU of its expectation.

Note: 1° $\bar{X}$, S^2 are functions of complete, sufficient statistic

(Luck) $(T_1(\underline{X}), T_2(\underline{X})) = (\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ for $\theta = (\mu, \sigma^2)$,

given $X \sim N(\mu, \sigma^2)$.

Hence, $\bar{X}$ and S^2 are UMVU for μ and σ^2 resp.

(Luck) 2° $\mathbb{E}_\theta(\bar{X}^2 - \frac{S^2}{n}) = \mu^2 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = \mu^2$

Hence $\bar{X}^2 - \frac{S^2}{n}$ is UMVU for μ^2.

(Luck) 3° $X_{(n)}$ is a complete sufficient statistic for θ, given $X \sim U(0, \theta)$.

Since $\mathbb{E}_\theta \frac{n+1}{n} X_{(n)} = \theta$, then $\frac{n+1}{n} X_{(n)}$ is UMVU for θ.

(Luck) 4° $\sum_{i=1}^n X_i$ is a complete sufficient statistic for θ, given $X \sim Bernoulli(\theta)$.

Since $\mathbb{E}_\theta \frac{1}{n} \sum_{i=1}^n X_i = \theta$, then $\frac{1}{n} \sum_{i=1}^n X_i$ is UMVU for θ.

(Solve) 5° (As in 4°) Look for UMVU estimator for $Var X = \theta(1-\theta)$.

Let $\mathbb{E}_\theta\, g(\sum_{i=1}^n X_i) = \theta(1-\theta)$, then

$$\sum_{k=0}^n g(k) \binom{n}{k} \theta^k (1-\theta)^{n-k} = \theta(1-\theta).$$

substitute $\eta = \frac{\theta}{1-\theta}$, we get $\sum_{k=0}^n g(k) \binom{n}{k} \eta^k = \eta(1+\eta)^{n-2}$

By comparing parameters, we find $g(k) = \frac{k(n-k)}{n(n-1)}$ $(k=0,\cdots,n)$

(Conditional) 6° $X_1, \cdots, X_n$ iid Exponential (λ),

$\sum_{i=1}^{n} X_i$ is a complete sufficient statistic for λ,

Look for UMVU estimator for $q(\lambda) = e^{-\lambda t}$.

Since $\mathbb{E}\,\mathbb{1}(X_1 > t) = P(X_1 > t) = e^{-\lambda t}$, $\mathbb{1}(X_1 > t)$ is an unbiased estimator of $q(\lambda)$.

Then $\mathbb{E}\left(\mathbb{1}(X_1>t) \,\middle|\, \sum_{i=1}^{n} X_i\right)$ is UMVU for $q(\lambda)$

$$\because \quad \mathbb{E}\left(\mathbb{1}(X_1>t) \,\middle|\, \sum_{i=1}^{n} X_i = s\right)$$

$$= P\left(X_1 > t \,\middle|\, \sum_{i=1}^{n} X_i = s\right)$$

$$= P\left(\frac{X_1}{\sum_{i=1}^{n} X_i} > \frac{t}{s} \,\middle|\, \sum_{i=1}^{n} X_i = s\right)$$

$$= P\left(\frac{X_1}{\sum_{i=1}^{n} X_i} > \frac{t}{s}\right) \qquad \left(\because \left(\frac{S_1}{S_n}, \cdots, \frac{S_{n-1}}{S_n}\right) \perp\!\!\!\perp S_n\right)$$

$$= P\left(U_{(1,n-1)} > \frac{t}{s}\right) \qquad \left(\because \left(\frac{S_1}{S_n}, \cdots, \frac{S_{n-1}}{S_n}\right) \sim \left(U_{(1)}, \cdots, U_{(n-1)}\right)\right)$$

$$= \left(1 - \frac{t}{s}\right)^{n-1}$$

$$\therefore \quad \left(1 - \frac{t}{\sum_{i=1}^{n} X_i}\right)^{n-1} \text{ is UMVU for } q(\lambda) = e^{-\lambda t}.$$

Note: $\left(1 - \frac{t}{\sum_{i=1}^{n} X_i}\right)^{n-1} \xrightarrow{P} e^{-\lambda t}$, so it's consistent.