Hamilton's Law & Lagrange's equation. ①

1、action integral : $S_{\text{(also called } J)} = \int_{t_1}^{t_2} L \, dt.$

where Lagrangian : $L = T - V.$

Hamilton's Law says : $\delta S(\underline{x}) = 0$

2、Lagrange's equation 在一维情况下的导出：

$$S = \int \left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x)\right] dt.$$

~~so $\delta S = \int_{t_1}^{t_2}$~~

so
$$S(\underline{x}+\eta) = \int \left[\frac{m}{2}\left(\frac{dx}{dt}+\frac{d\eta}{dt}\right)^2 - V(\underline{x}+\eta)\right] dt.$$

$$= \int \left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + m\frac{dx}{dt}\frac{d\eta}{dt} - \left(V(\underline{x}) + \eta V'(\underline{x})\right) + \text{(second and higher order)}\right] dt$$

$$\therefore \delta S = \int_{t_1}^{t_2} \left[m\frac{dx}{dt}\frac{d\eta}{dt} - \eta V'(\underline{x})\right] dt.$$

with integration by parts , $\int_{t_1}^{t_2} m\frac{dx}{dt}\frac{d\eta}{dt} dt = m\frac{dx}{dt}\eta\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta \cdot m\frac{d^2x}{dt^2} dt$

where $m\frac{dx}{dt}\eta\Big|_{t=t_1,t_2} = 0$, for $\eta\Big|_{t=t_1,t_2} \equiv 0.$

$$\therefore \delta S = \int_{t_1}^{t_2} \left(-m\frac{d^2x}{dt^2} - V'(x)\right)\eta \, dt \equiv 0$$

根据变分法基本引理，

$$-m\frac{d^2x}{dt^2} - V'(\underline{x}) = 0$$

notes: 1. here $-V'(\underline{x}) = F$, so the result is just $F = ma$

2、在三维情况下也可类似的导出，但前提条件是所有坐标相互独立

3、note 1 simply showed the equivalence of Lagrange's Equation and Newton's second law, hence, all the derivation above showed these relationships :

Hamilton's Law $\Rightarrow$ Lagrange's Equation $\Leftrightarrow$ Newton's second law.

Only till now, Hamilton's **Law** has theoretical support — Newton's second law, and no longer an artificial claim.

Derivation from Hamilton's Law to Lagrange's Equation for conservative systems.

First, we have Hamilton's Law,

$$\delta S = \delta\int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2}\delta L\,dt = \int_{t_1}^{t_2}\sum_i\left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta\dot{q}_i\right)dt = 0$$

where L has the form $L(q_i, \dot{q}_i)$, and q_i is a generalized coordinate.
note that δq_i means the difference between two slightly differing functions of time t, and we have the equivalence:

$$\delta\dot{q}_i = \frac{d}{dt}\delta q_i$$

Next, with integration by parts,

$$\int_{t_1}^{t_2}\sum_i\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}(\delta q_i)\,dt = \sum_i\frac{\partial L}{\partial \dot{q}_i}\delta q_i\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\sum_i(\delta q_i)\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\,dt.$$

for $\delta q_i = 0$ when $t = t_1, t_2$, so the first term vanishes.
now we have,

$$\int_{t_1}^{t_2}\sum_i\left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right]\delta q_i\,dt = 0$$

For a conservative system, each generalized coordinate q_i is independent of the others, as is each variation δq_i.
With fundamental lemma of calculus of variations, we have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \quad i = 1, 2, \cdots, n. \quad \square$$

We are wild, exciting, challenging, fun and excellent.

Derivation from Newton's second law to Lagrange's Equation.

We do not consider constraint forces in the derivation below, and hence all the coordinates are independent from each other. Let $\vec{F} = \vec{F}^c + \vec{F}^{nc}$ (conservative and nonconservative)

First we have Newton's second law in an inertial Cartesian coordinate system,

$$m\ddot{x}_1(t) = F_1$$
$$m\ddot{x}_2(t) = F_2$$
$$m\ddot{x}_3(t) = F_3$$

Next, transform the Cartesian coordinates to a set of generalized coordinates q_1, q_2, q_3, and they are independent from each other. The transform functions are:

$$x_1(q_1, q_2, q_3), \quad x_2(q_1, q_2, q_3), \quad x_3(q_1, q_2, q_3).$$

Here goes some preliminary steps.

(a) $\dot{x}_1(t) = \frac{\partial x_1}{\partial q_1}\dot{q}_1 + \frac{\partial x_1}{\partial q_2}\dot{q}_2 + \frac{\partial x_1}{\partial q_3}\dot{q}_3$

hence $\dot{x}_1$ can be written as $\dot{x}_1(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$, notice that the six components are independent.

so $\frac{\partial \dot{x}_1}{\partial \dot{q}_1} = \frac{\partial x_1}{\partial q_1}, \quad \frac{\partial \dot{x}_1}{\partial \dot{q}_2} = \frac{\partial x_1}{\partial q_2}, \quad \frac{\partial \dot{x}_1}{\partial \dot{q}_3} = \frac{\partial x_1}{\partial q_3}$ (1)

(b)
$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}T = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}\left[\frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\right]$$
$$= \frac{d}{dt}\left[m\left(\dot{x}_1\frac{\partial x_1}{\partial q_1} + \dot{x}_2\frac{\partial x_2}{\partial q_1} + \dot{x}_3\frac{\partial x_3}{\partial q_1}\right)\right], \text{ (use (1))}$$
$$= m\left(\ddot{x}_1\frac{\partial x_1}{\partial q_1} + \ddot{x}_2\frac{\partial x_2}{\partial q_1} + \ddot{x}_3\frac{\partial x_3}{\partial q_1}\right) + m\left(\dot{x}_1\frac{d}{dt}\frac{\partial x_1}{\partial q_1} + \dot{x}_2\frac{d}{dt}\frac{\partial x_2}{\partial q_1} + \dot{x}_3\frac{d}{dt}\frac{\partial x_3}{\partial q_1}\right)$$

now we show $\frac{d}{dt}\frac{\partial x_i}{\partial q_1} = \frac{\partial}{\partial q_1}\frac{dx_i}{dt}$

$$\frac{d}{dt}\frac{\partial x_i}{\partial q_1} = \frac{\partial}{\partial q_1}\left(\frac{\partial x_i}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial x_i}{\partial q_1}\right)\dot{q}_2 + \frac{\partial}{\partial q_3}\left(\frac{\partial x_i}{\partial q_1}\right)\dot{q}_3$$

$$\frac{\partial}{\partial q_1}\frac{dx_i}{dt} = \frac{\partial}{\partial q_1}\left(\frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\dot{q}_2 + \frac{\partial x_i}{\partial q_3}\dot{q}_3\right)$$
$$= \frac{\partial}{\partial q_1}\left(\frac{\partial x_i}{\partial q_1}\right)\dot{q}_1 + \frac{\partial}{\partial q_1}\left(\frac{\partial x_i}{\partial q_2}\right)\dot{q}_2 + \frac{\partial}{\partial q_1}\left(\frac{\partial x_i}{\partial q_3}\right)\dot{q}_3. \ \square$$ (for q_i and $\dot{q}_j$ are independent)

hence,
$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}T = \left(F_1\frac{\partial x_1}{\partial q_1} + F_2\frac{\partial x_2}{\partial q_1} + F_3\frac{\partial x_3}{\partial q_1}\right) + \frac{\partial T}{\partial q_1}$$
$$= -\frac{\partial V}{\partial q_1} + \frac{\partial T}{\partial q_1} + \left(F_1^{nc}\frac{\partial x_1}{\partial q_1} + F_2^{nc}\frac{\partial x_2}{\partial q_1} + F_3^{nc}\frac{\partial x_3}{\partial q_1}\right). \quad (2)$$

(c) for V is in the form of $V(q_1, q_2, q_3)$, then $\frac{\partial V}{\partial \dot{q}_i} = 0$.

hence $\frac{d}{dt}\frac{\partial V}{\partial \dot{q}_1} = 0$ (3)

Now we add (2), (3), and get.

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_1}(T-V) - \frac{\partial}{\partial q_1}(T-V) = F_1^{nc}\frac{\partial x_1}{\partial q_1} + F_2^{nc}\frac{\partial x_2}{\partial q_1} + F_3^{nc}\frac{\partial x_3}{\partial q_1}$$

Finally, we attained Lagrange's Equation in generalized coordinates (q_1, q_2, q_3)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = F_1^{nc}\frac{\partial x_1}{\partial q_i} + F_2^{nc}\frac{\partial x_2}{\partial q_i} + F_3^{nc}\frac{\partial x_3}{\partial q_i} \quad (i = 1, 2, 3).$$

specially, when $\vec{F}$ is conservative, we have.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad . \square$$

note: for the derivation suits any generalized coordinates, the derivation above also attained Lagrange's equation for Cartesian Coordinates.

Application of Lagrange's Equation in constrained systems. ④

1. Recipe when using the Lagrange's Equation when constraints are involved.
   (a) Reduce the three Cartesian coordinates to a smaller number of independent generalized coordinates according to constraint condition.
   (b) Express Lagrangian in terms of generalized coordinates.
   (c) Apply the Lagrange's Equations to each of the generalized coordinates to obtain the equation of motion.

   note: the Cartesian frame choosed in step (a) must be an inertial frame.

2. Proof of Lagrange's Equation with constrains.

First, we prove that Hamilton's principle still holds on constrained region.

we denote $\vec{R}(t)$ is a neighboring "wrong" path compared to $\vec{r}(t)$, the right path, and

$\vec{R}(t) = \vec{r}(t) + \vec{\varepsilon}(t)$, $\vec{\varepsilon}(t)$ is an infinitesimal vector from $\vec{\varepsilon}(t)$ to $\vec{R}(t)$. note that, $\vec{R}(t)$ and $\vec{r}(t)$ are now both on the constrained region, and the degree of freedom of $\vec{\varepsilon}(t)$ is no longer 3.

we can get $\delta S = -\int_{t_1}^{t_2} \vec{\varepsilon} \cdot [m\ddot{\vec{r}} + \nabla U]\, dt$ without any difference from the former one. and now

$m\ddot{\vec{r}} = \vec{F}_{tot} = \vec{F}_{cstr} + \vec{F}$, $\nabla U = -\vec{F}$

$\therefore \delta S = -\int_{t_1}^{t_2} \vec{\varepsilon} \cdot \vec{F}_{cstr}\, dt$. for $\vec{F}_{cstr}$ is normal to the constrained region while $\vec{\varepsilon}$ is just in the constrained region, $\vec{\varepsilon} \cdot \vec{F}_{cstr} \equiv 0$

$\therefore \delta S \equiv 0$ for any $\vec{\varepsilon}$ on the constrained region

Next, from Hamilton's principle, we can prove Lagrange's Equations with respect to the appropriate generalized coordinates. However, we cannot prove them in three Cartesian coordinates, which, actually, we don't need.

Conclusion: For any holonomic system, with n degrees of freedom and n generalized coordinates, and with the nonconstraint forces derivable from a potential energy $U(q_1, q_2, \cdots, q_n, t)$, the path followed by the system is determined by the n Lagrange's Equations.

Derivation of Lagrange's Equations from D'Alembert's Principle

⑤

1. D'Alembert's Principle

define: $\underline{F}_{loss} \equiv \underline{F} - m\underline{a}$, where $\underline{F}$ is the total force except constraint force

then, D'Alembert's Principle is

$$\underline{F}_{loss} + \underline{F}_{cstr} = 0 \qquad (1^\circ)$$

Actually, this is an another form of Newton's Second Law.

2. Lagrange-d'Alembert's Principle

For system of N particles, $\underline{F}_{cstr_i}$ is the constraint force on the ith particle.

Suppose all the constraints are ideal, which means,

$$\sum_{i=1}^{n} \underline{F}_{cstr_i} \cdot \delta \underline{r}_i = 0, \qquad (2^\circ)$$

where $\delta \underline{r}_i$ is a virtual displacement of the i-th particle.

from $(1^\circ), (2^\circ)$, we get Lagrange-d'Alembert's Principle.

$$\sum_{i=1}^{n} (\underline{F}_i - m_i \underline{a}_i) \cdot \delta \underline{r}_i = 0$$

3. Derivation of Lagrange's Equation from Lagrange-d'Alembert's Principle.

(A) For a system of N particles under ideal constrains,

$s = 3n - k$, is the degree of freedom of the system, k is the number of constraints.

The displacement vector of the i-th particle is.

$$\underline{r}_i = \underline{r}_i(q_1, q_2, \cdots, q_s, t)$$

where $q_1, q_2, \cdots, q_s$ are generalized coordinates.

Velocity is $\underline{v}_i = \frac{d\underline{r}_i}{dt} = \sum_{k=1}^{s} \frac{\partial \underline{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \underline{r}_i}{\partial t}$

$$\therefore \begin{cases} \dfrac{\partial \underline{v}_i}{\partial \dot{q}_j} = \dfrac{\partial \underline{r}_i}{\partial q_j} & (3^\circ) \\[2ex] \dfrac{\partial \underline{v}_i}{\partial q_j} = \displaystyle\sum_{k=1}^{s} \dfrac{\partial^2 \underline{r}_i}{\partial q_j \partial q_k} \dot{q}_k + \dfrac{\partial^2 \underline{r}_i}{\partial q_j \partial t} = \dfrac{d}{dt}\left(\dfrac{\partial \underline{r}_i}{\partial q_j}\right) & (4^\circ) \end{cases}$$

Virtual Displacement is

$$\delta \underline{r}_i = \sum_{j=1}^{s} \frac{\partial \underline{r}_i}{\partial q_j} \delta q_j \qquad (5^\circ)$$

and $\sum_{i=1}^{n} m_i \underline{a}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j} = \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\underline{v}_i \cdot \frac{\partial \underline{r}_i}{\partial q_j}\right) - \sum_{i=1}^{n} m_i \underline{v}_i \cdot \frac{d}{dt}\left(\frac{\partial \underline{r}_i}{\partial q_j}\right)$ (integration by pa

$$= \sum_{i=1}^{n} m_i \frac{d}{dt}\left(\underline{v}_i \cdot \frac{\partial \underline{v}_i}{\partial \dot{q}_j}\right) - \sum_{i=1}^{n} m_i \underline{v}_i \cdot \frac{\partial \underline{v}_i}{\partial q_j} \qquad ((3^\circ), (4^\circ))$$

$$= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \sum_{i=1}^{n} \frac{1}{2} m_i \underline{v}_i \cdot \underline{v}_i\right) - \frac{\partial}{\partial q_j}\left(\sum_{i=1}^{n} \frac{1}{2} m_i \underline{v}_i \cdot \underline{v}_i\right)$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \qquad (6^\circ)$$

From Lagrange–d'Alembert's Principle,

$$\sum_{i=1}^{n}\left[\underline{F_i}\cdot\left(\sum_{j=1}^{s}\frac{\partial r_i}{\partial q_j}\delta q_j\right) - m_i\,\underline{a_i}\cdot\left(\sum_{j=1}^{s}\frac{\partial r_i}{\partial q_j}\delta q_j\right)\right] = 0$$

$$\therefore \quad \sum_{j=1}^{s}\left[\left(\sum_{i=1}^{n}\underline{F_i}\cdot\frac{\partial r_i}{\partial q_j}\right)\delta q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\delta q_j\right] = 0$$

define $Q_j = \sum_{i=1}^{n}\underline{F_i}\cdot\frac{\partial r_i}{\partial q_j}$ $(j=1,2,\cdots,s)$, as Generized Forces of a system of particles.

Then $\sum_{j=1}^{s}\left[Q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)\right]\delta q_j = 0$

for δq_j are independent from each other,

we get <u>Lagrange's Equation of the second kind</u>,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad (j=1,2,\cdots,s)$$

notes: 1. $\sum_{j=1}^{s} Q_j\,\delta q_j = \sum_{i=1}^{n}\underline{F_i}\cdot\delta \underline{r_i}$

2. if all $\underline{F_i}$ are conservative and have a total potential $V(q_1, q_2, \cdots q_s, t)$

the $\sum_{i=1}^{n}\underline{F_i}\cdot\delta \underline{r_i} = -\delta V = -\sum_{j=1}^{s}\frac{\partial V}{\partial q_j}\delta q_j$

hence, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$

(B) substitute $q_1, q_2, \cdots, q_s$ with $x_1, x_2, \cdots, x_{3n}$, which are not independent from each other. and not have to be Cartesian coordinates.

then displacement is in the form, $\underline{r_i} = \underline{r_i}(x_1, x_2, \cdots, x_{3n})$

similarly, we can derive

$$\sum_{j=1}^{3n}\left[Q_j - \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j}\right)\right]\delta x_j = 0 \quad (j=1,2,\cdots,3n) \qquad (\divideontimes)$$

while δx_j are not independent from each other, $Q_j = \sum_{i=1}^{n}\underline{F_i}\cdot\frac{\partial r_i}{\partial x_j}$.

For $\delta f_i = \sum_{j=1}^{3n}\frac{\partial f_i}{\partial x_j}\delta x_j = 0 \quad (i=1,2,\cdots,k)$ (Constraints)

are independent from eachother, at any time (or position), we can find λ_i s.t.

$$Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) + \frac{\partial T}{\partial x_j} + \sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial x_j} = 0 \quad (j = 3n-k+1, 3n-k+2, \cdots, 3n) \qquad (a)$$

So $(\divideontimes) + \sum_{i=1}^{k}\lambda_i\delta f_i$, is

$$\sum_{j=1}^{3n}\left[Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) + \frac{\partial T}{\partial x_j} + \sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial x_j}\right]\delta x_j = 0$$

from the independence of δx_j $(j=1,2,\cdots,s)$

$$Q_j - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) + \frac{\partial T}{\partial x_j} + \sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial x_j} = 0 \quad (j=1,2,\cdots,s) \qquad (b)$$

combine (a), (b), we have <u>Lagrange's Equation of the first kind</u>.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_j}\right) - \frac{\partial T}{\partial x_j} = Q_j + \sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial x_j} \quad (j=1,2,\cdots,3n)$$

Note: 1° we can <u>solve x_j, λ_i as a function of t</u> with the equations above and those k constraint functions f_i; 2° $\sum_{i=1}^{k}\lambda_i\frac{\partial f_i}{\partial x_j} = \sum_{i=1}^{n}\underline{F_{cstr_i}}\cdot\frac{\partial r_i}{\partial x_j}$, this the physical interpretation of the new term and how we get F_{cstr} from λ_i