Sp. I

## Asymptotic order statistics

(1) (Uniform Case) For $E \sim$ Exponential(1), ~~Sample mean $S_k \sim Ex$~~

Sample sum has limit distribution

$$\sqrt{k}\left(\frac{S_k}{k} - 1\right) \xrightarrow{d} N(0,1)$$

Let $0 < p_1 < p_2 < 1$, and $k_1 = \lfloor np_1 \rfloor$, $k_2 = \lfloor np_2 \rfloor$,

then $\sqrt{n+1}\left(\frac{S_{k_1}}{n+1} - \frac{k_1}{n+1}\right) = \frac{\sqrt{k_1}}{\sqrt{n+1}} \cdot \sqrt{k_1}\left(\frac{S_{k_1}}{k_1} - 1\right)$

$$\xrightarrow{d} \sqrt{p_1}\, N(0,1)$$

$$\sim N(0, p_1)$$

Similarly, $\sqrt{n+1}\left[\frac{(S_{k_2} - S_{k_1})}{n+1} - \frac{k_2 - k_1}{n+1}\right] \xrightarrow{d} N(0, p_2 - p_1)$

$$\sqrt{n+1}\left[\frac{S_{n+1} - S_{k_2}}{n+1} - \frac{n+1-k_2}{n+1}\right] \xrightarrow{d} N(0, 1-p_2)$$

~~$\because \forall n \in \mathbb{R}$,~~

~~$S_{k_1}$, $(S_{k_2} - S_{k_1})$, $(S_{n+1} - S_{k_2})$ are mutually independent.~~

Denote

$X_{1,n} \equiv \sqrt{n}\left(\frac{S_{k_1}}{n+1} - p_1\right)$, $X_{2,n} \equiv \sqrt{n}\left[\frac{S_{k_2} - S_{k_1}}{n+1} - (p_2 - p_1)\right]$,

$X_{3,n} \equiv \sqrt{n}\left[\frac{S_{n+1} - S_{k_2}}{n+1} - (1 - p_2)\right]$,

then it's easy to see that

$$X_{1,n} \xrightarrow{d} N(0, p_1), \quad X_{2,n} \xrightarrow{d} N(0, p_2 - p_1), \quad X_{3,n} \xrightarrow{d} N(0, 1-p_2)$$

$\because \forall n \in \mathbb{N}$, $X_{1,n}$, $X_{2,n}$, $X_{3,n}$ are mutually independent.

$\therefore \underline{X}_n \equiv (X_{1,n}, X_{2,n}, X_{3,n})^T \xrightarrow{d} N(0, p_1) \cdot N(0, p_2 - p_1) \cdot N(0, 1-p_2)$

$$\sim N(\underline{0}, \mathrm{diag}(p_1, p_2 - p_1, 1 - p_2))$$

From order statistics, we know

$$\left(\frac{S_1}{S_{n+1}}, \cdots, \frac{S_n}{S_{n+1}}\right) \sim (U_{(1)}, \cdots, U_{(n)})$$

Construct $g(x_1, x_2, x_3) = \left(\frac{x_1}{x_1+x_2+x_3}, \frac{x_1+x_2}{x_1+x_2+x_3}\right)^T$,

then ~~$g(X_n) = \left(\frac{S_{k_1}}{S_{n+1}}, \frac{S_{k_2}}{S_{n+1}}\right)^T$~~

$$g\left(\frac{S_{k_1}}{n+1}, \frac{S_{k_2}-S_{k_1}}{n+1}, \frac{S_{n+1}-S_{k_2}}{n+1}\right) = \left(\frac{S_{k_1}}{S_{n+1}}, \frac{S_{k_2}}{S_{n+1}}\right)^T$$

$$\sim (U_{(k_1)}, U_{(k_2)})^T$$

$$g(p_1, p_2-p_1, 1-p_2) = (p_1, p_2)^T$$

Using delta method,

$$\sqrt{n}\left[g\left(\frac{S_k}{n+1}, \frac{S_{k_2}-S_{k_1}}{n+1}, \frac{S_{n+1}-S_{k_2}}{n+1}\right) - g(p_1, p_2-p_1, 1-p_2)\right]$$

$$\xrightarrow{d} \dot{g}^T(p_1, p_2-p_1, 1-p_2)\, N\left(\underline{0}, \begin{pmatrix} p_1 & & \\ & p_2-p_1 & \\ & & 1-p_2 \end{pmatrix}\right)$$

$$\sim N\left(\underline{0}, \begin{pmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{pmatrix}\right).$$

Notice that $\dot{g}^T(\underline{x})$ is required to be continuous at $\underline{x} = (p_1, p_2-p_1, 1-p_2)^T$, but it's obvious.

Hence, we have

$$\sqrt{n}\left[\begin{pmatrix} U_{(k_1)} \\ U_{(k_2)} \end{pmatrix} - \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right] \xrightarrow{d} N\left(\underline{0}, \begin{pmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{pmatrix}\right).$$

$$(0 < p_1 < p_2 < 1)$$

(2) ~~[scribbled out]~~

General case

$X \sim F$ , $F$ is continuous and strictly increasing.

Integral transformation gives $F(X) \sim U$, and $F^{-1}(U) \sim X$.

(Random) Samples $(X_1, \cdots, X_n) \sim (F^{-1}(U_1), \cdots, F^{-1}(U_n))$

Order statistics $(X_{(1)}, \cdots, X_{(n)}) \sim (F^{-1}(U_{(1)}), \cdots, F^{-1}(U_{(n)}))$,

Using the monotonicity of $F^{-1}(x)$.

Let $g(x_1, x_2) = (F^{-1}(x_1), F^{-1}(x_2))^T$,

then for $0 < p_1 < p_2 < 1$,

$$\dot{g}(p_1, p_2) = \begin{pmatrix} \frac{1}{f(x_{p_1})} & \\ & \frac{1}{f(x_{p_2})} \end{pmatrix}, \quad \text{where } \begin{cases} F(x_{p_1}) = p_1 \\ F(x_{p_2}) = p_2 \end{cases}.$$

$$g(p_1, p_2) = (x_{p_1}, x_{p_2})^T$$

Using the result of (1) and delta method, we have

$$\sqrt{n}\left[\begin{pmatrix} X_{(k_1)} \\ X_{(k_2)} \end{pmatrix} - \begin{pmatrix} x_{p_1} \\ x_{p_2} \end{pmatrix}\right] \xrightarrow{d} N(0, \dot{g}^T(p_1, p_2)\, \Sigma\, \dot{g}(p_1, p_2))$$

where $\Sigma = \begin{pmatrix} p_1(1-p_1) & p_1(1-p_2) \\ p_1(1-p_2) & p_2(1-p_2) \end{pmatrix}$

We calculate that

$$\Sigma_X = \dot{g}^T(p_1, p_2) \cdot \Sigma \cdot \dot{g}(p_1, p_2) = \begin{pmatrix} \frac{p_1(1-p_1)}{f(x_{p_1})^2} & \frac{p_1(1-p_2)}{f(x_{p_1})f(x_{p_2})} \\ \frac{p_1(1-p_2)}{f(x_{p_1})f(x_{p_2})} & \frac{p_2(1-p_2)}{f(x_{p_2})^2} \end{pmatrix}.$$

E.g.: 1° $X \sim N(\mu, \sigma^2)$, then

$$\sqrt{n}\left(X_{([\frac{n}{2}])} - x_{\frac{1}{2}}\right) \xrightarrow{d} N(0, \tfrac{\pi}{2}\sigma^2)$$

Comparing to $\sqrt{n}(\overline{X} - x_{\frac{1}{2}}) \xrightarrow{d} N(0, \sigma^2)$

We lost efficiency.

Asymptotic relative efficiency : $\frac{\frac{\pi}{2}\sigma^2}{\sigma^2} = \frac{\pi}{2} > 1$.

2° $X \sim$ Cauchy$(\mu, \sigma)$, then

$$\sqrt{n}\left[\left(X_{([\frac{3}{4}])} - X_{([\frac{1}{4}])}\right) - \left(x_{\frac{3}{4}} - x_{\frac{1}{4}}\right)\right] \xrightarrow{d} N(0, \pi^2\sigma^2).$$

~~$\therefore x_{\frac{3}{4}} = 1,\ x_{\frac{1}{4}} = -1$ for Cauchy$(0,1)$~~

Details:

For Cauchy$(0,1)$, $x_{\frac{3}{4}} = 1$, $x_{\frac{1}{4}} = -1$,

For Cauchy$(\mu,\sigma)$, $x_{\frac{3}{4}} = \mu + \sigma$, $x_{\frac{1}{4}} = \mu - \sigma$.

$$\Sigma = \begin{pmatrix} \frac{3}{16} & \frac{1}{16} \\ \frac{1}{16} & \frac{3}{16} \end{pmatrix}, \quad f(x_{\frac{3}{4}}) = f(x_{\frac{1}{4}}) = \frac{1}{2\pi\sigma}$$

$$\Sigma_X = \begin{pmatrix} \frac{3}{4}\pi^2\sigma^2 & \frac{1}{4}\pi^2\sigma^2 \\ \frac{1}{4}\pi^2\sigma^2 & \frac{3}{4}\pi^2\sigma^2 \end{pmatrix} \quad \text{for } \left(X_{([\frac{1}{4}])}, X_{([\frac{3}{4}])}\right)^T.$$

$$\therefore \Sigma_X = (-1, 1)\begin{pmatrix} \frac{3}{4}\pi^2\sigma^2 & \frac{1}{4}\pi^2\sigma^2 \\ \frac{1}{4}\pi^2\sigma^2 & \frac{3}{4}\pi^2\sigma^2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \pi^2\sigma^2 \text{ for } X_{([\frac{3}{4}])} - X_{([\frac{1}{4}])}$$

(Again, delta method).