# Part A: Banach Spaces

**Def.**: For a linear space $X$, function $\|\cdot\|: X \to \mathbb{R}$ is a norm if

(1a) Nonnegativity $\|x\| \geq 0$

(1b) Positive definiteness $\|x\| = 0 \leftrightarrow x = 0$

(2) Triangle inequality $\|x+y\| \leq \|x\| + \|y\|$

(3) Homgeneity $\|\alpha x\| = |\alpha| \, \|x\|$.

**Def**: Normed linear space $(X, \|\cdot\|)$

Note: $d(x,y) = \|x-y\|$ is the metric derived from the norm $\|\cdot\|$.

**Def**: Banach space : complete normed (linear) space.

**Thm**: Series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent $\xRightarrow{X \text{ is a Banach space}}$ it is unconditionally convergent.

Note: Def: $1^\circ$ absolutely convergent : $\sum_{n=1}^{\infty} \|x_n\|$ is convergent

$2^\circ$ unconditionally convergent : Any reordering of the series converges to a same limit.

**Thm**: $M$ is a linear subspace of a normed linear space $X$

$1^\circ$ $M$ is open $\Rightarrow M = X$

$2^\circ$ $\overline{M}$ is a closed linear subspace

$3^\circ$ $M$ is a Banach space $\xLeftrightarrow{X \text{ is Banach}}$ $M$ is closed.

**Thm**: (Riesz)

$M$ is a proper closed linear subspace of a normed linear space $X$, then $d(x, M) \leq \|x\|$, but the inequality may not be achieved.

**Def:** <u>Continuous linear transformation</u> : $L: X \to Y$ linear and continuous.
$X, Y$ are normed linear spaces.

**Thm:** $L: X \to Y$, $X$ and $Y$ are normed linear space.
(Principle of Superposition)

$L$ is a linear continuous trasformation $\iff$ $L\left(\sum_{i=1}^{\infty} \alpha_i x_i\right) = \sum_{i=1}^{\infty} \alpha_i L(x_i)$,
$\forall$ convergent series $\sum_{i=1}^{\infty} \alpha_i x_i$.

**Def:** A linear transformation between two normed linear spaces $L: X \to Y$

$1^\circ$ <u>bounded</u>, if

$$\exists M \geq 0, \ \forall x \in X, \ \|Lx\| \leq M\|x\|.$$

$2^\circ$ <u>bounded below</u>, if

$$\exists m > 0, \ \forall x \in X, \ \|Lx\| \geq m\|x\|$$

**Thm:** A linear transformation between two normed linear spaces,
continuous $\iff$ bounded.

**Lemma:** A linear transformation between two normed linear spaces,
continuous at one point $\Longrightarrow$ continuous everywhere.

**Thm:** $L: X \to Y$ is a continuous linear transformation $\Longrightarrow$
The null space $N(L)$ is a closed linear subspace of $X$.

**Thm:** A linear transformation between two normed linear spaces,
bounded below $\iff$ exists a continuous inverse on its range.

**Def:** <u>Norm of bounded linear transformation (Blt)</u> is the supremum of the transformation
(Operator norm)

$$\|T\| \equiv \inf\{M : \|Tx\| \leq M\|x\|, \ \forall x \in X\}.$$

**Lemma:** (Alternative definitions of norm of Blt)

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|=1} \|Tx\| = \sup_{\|x\| \neq 0} \frac{\|Tx\|}{\|x\|}$$

~~Thm: $\|\cdot\|$ is a norm on the space of bounded linear transformations between two normed linear spaces.~~

Thm: $(Blt[X, Y], \|\cdot\|)$ is a normed linear space.

Thm: $T \in Blt[X,Y]$, $S \in Blt[Y,Z] \Rightarrow ST \in Blt[X,Z]$, $\|ST\| \leq \|S\| \cdot \|T\|$

Def: Convergence in the operator norm topology (Convergence in the uniform norm topology):

A sequence of Blt converges uniformly if it converges in operator norm $\|\cdot\|$.

Thm: $(Blt[X,Y], \|\cdot\|)$ is a Banach space $\Leftarrow$ $Y$ is a Banach space.

Thm: A BLT from a dense linear subspace of a normed linear space into a Banach space can be uniquely extended to a BLT on the whole space; They have the same operator norm.

Def: Strong convergence of BLT sequences:

A sequence $\{T_n\}$ of BLT converges strongly to $T \in Blt[X,Y]$, if $\{T_n x\}$ converges to $Tx$ for every $x \in X$.

Lemma: A BLT sequence,

converges uniformly $\Rightarrow$ converges strongly.

Def: 1° Topologically isomorphic: exists continuous linear transformations $\phi$ and $\phi^{-1}$ between normed linear spaces: $X \underset{\phi^{-1}}{\overset{\phi}{\rightleftarrows}} Y$

(Topological isomorphism): $\phi$.

2° Isometrically isomorphic: $\|\phi x\| = \|x\|$, $\forall x \in X$.

(Isometric isomorphism): $\phi$: isomorphism that preserves norm

Thm: $X, Y$ are two normed linear spaces, they are topologically isomorphic $\Longleftrightarrow$ $\exists$ linear $\phi$ from $X$ onto $Y$ which is bounded and bounded below.

Def: <u>Equivalent norms</u> : generated metrics are equivalent.

Cor: On a linear space,

two norms are equivalent $\Longleftrightarrow$ ratio of the norms are bounded and bounded below.

Lemma: For a given basis on a finite-dimensional normed linear space, the expansion coefficients are continuous linear functions on the space.

Thm: 1° Finite-dimensional normed linear spaces are always complete.

2° Finite-dimensional linear subspaces of a normed linear space are always closed.

3° Linear transformations on a finite-dimensional normed linear space are always continuous.

4° finite-dimensional normed linear spaces

isomorphic $\Rightarrow$ topologically isomorphic $\Longleftrightarrow$ equi-dimensional.

5° Norms on a finite-dimensional linear space are always equivalen

6° $X$ is a normed linear space,

$X$ is finite-dimensional $\Longleftrightarrow$ $\{x \in X : \|x\| \leq 1\}$ is compact.