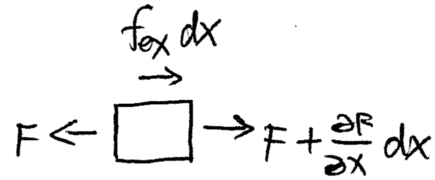
For B.C.:

$$\boxed{\begin{aligned} M_z &= EI_z v'' \\ -Q_y &= (M_z)' = (EI_z v'')' \\ q &= -(Q_y)' = (EI_z v'')'' \end{aligned}}$$

V. Continuous

1. Longitudinal

$$\mu\, dx \frac{\partial^2 u}{\partial t^2} = \left[\left(F + \frac{\partial F}{\partial x} dx\right) + f_{ex}\, dx\right] - F$$

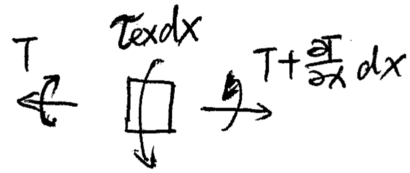


$$\Rightarrow \quad \frac{\partial F}{\partial x} - \mu \frac{\partial^2 u}{\partial t^2} + f_{ex} = 0$$

$$\because \quad \frac{F}{A} = E \frac{\partial u}{\partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) - \mu \frac{\partial^2 u}{\partial t^2} + f_{ex} = 0$$

2. Torsional



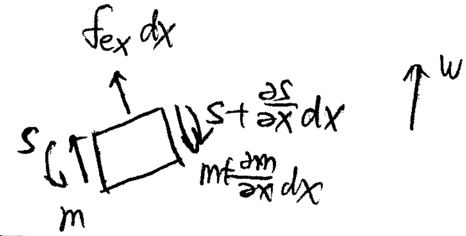
$$I\, dx \frac{\partial^2 \theta}{\partial t^2} = \left[\left(T + \frac{\partial T}{\partial x} dx\right) + \tau_{ex}\, dx\right] - T$$

$$\Rightarrow \quad \frac{\partial T}{\partial x} - I \frac{\partial^2 \theta}{\partial t^2} + \tau_{ex} = 0$$

$$\because \quad T = GJ \frac{\partial \theta}{\partial x}$$

$$\Rightarrow \quad \frac{\partial}{\partial x}\left(GJ \frac{\partial \theta}{\partial x}\right) - I \frac{\partial^2 \theta}{\partial t^2} + \tau_{ex} = 0$$

3. Flexural



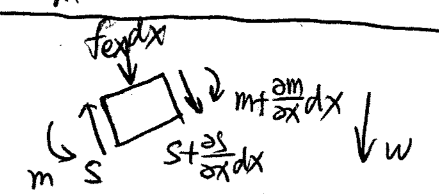
$$\mu\, dx \frac{\partial^2 w}{\partial t^2} = [S + f_{ex} dx] - \left[S + \frac{\partial S}{\partial x} dx\right]$$

$$\Rightarrow \quad -\frac{\partial S}{\partial x} - \mu \frac{\partial^2 w}{\partial t^2} + f_{ex} = 0$$

$$\because \quad \frac{\partial m}{\partial x} = -S$$

$$m = EI \frac{\partial^2 w}{\partial x^2}$$

$$\Rightarrow \quad + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) - \mu \frac{\partial^2 w}{\partial t^2} + f_{ex} = 0$$



$$\begin{cases} \mu\, dx \frac{\partial^2 w}{\partial t^2} = \left[\left(S + \frac{\partial S}{\partial x} dx\right) + f_{ex} dx\right] - S \\ \frac{\partial m}{\partial x} = -S \end{cases}$$

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) + \mu \frac{\partial^2 w}{\partial t^2} = f_{ex}$$

Transversal vibration solution:

(free vibration)

$$\frac{\partial^2}{\partial x^2}\left[EI(x) \frac{\partial^2 w(x,t)}{\partial x^2} \right] = -\mu(x) \frac{\partial^2 w(x,t)}{\partial t^2}$$

Using method of separation of variables,

$$w(x,t) = W(x) f(t)$$

$$\Rightarrow \quad \frac{1}{\mu(x) W(x)} \frac{d^2}{dx^2}\left[EI(x) \frac{d^2 W(x)}{dx^2} \right] = -\frac{1}{f(t)} \frac{d^2 f(t)}{dt^2} = \omega^2$$

$$\therefore \quad \begin{cases} \dfrac{d^2}{dx^2}\left[EI(x) \dfrac{d^2 W(x)}{dx^2} \right] - \omega^2 \mu(x) W(x) = 0 \\ \dfrac{d^2 f(t)}{dt^2} + \omega^2 f(t) = 0 \end{cases}$$

Denote
$$\begin{cases} L = \dfrac{d^2}{dx}\left[E I(x) \dfrac{d^2}{dx^2} \right] \\ m = \mu(x) \\ \lambda = \omega^2 \end{cases}$$

then $\quad L[W] = \lambda \, m[W]$

$W(x)$ is an eigenfunction of L associated with eigenvalue λ.

Orthogonality of Eigenfunctions:

If the problem is self-adjoint, i.e.

$$\begin{cases} \displaystyle\int_D W_s L[W_r] \, dD = \int_D W_r L[W_s] \, dD \\ \displaystyle\int_D W_s m[W_r] \, dD = \int_D W_r m[W_s] \, dD \end{cases}$$

where W_r and W_s are eigenfunctions associated with two distinct eigenvalues λ_r, λ_s.

$\because$ $L[W_r] = \lambda_r m[W_r]$

$L[W_s] = \lambda_s m[W_s]$

$\therefore$ $\int_D W_s L[W_r] - W_r L[W_s] \, dD = \int_D \lambda_r W_s m[W_r] - \lambda_s W_r m[W_s] \, dD$

$\therefore$ $0 = (\lambda_r - \lambda_s) \int_D W_r m[W_s] \, dD$

$\therefore$ $\int_D W_r m[W_s] \, dD = 0$ (Orthogonality)

$\therefore$ W_r and W_s are orthogonal.

Expansion Theorem:
(Orthonormal)

Eigenfunctions W_r form a complete set.

Any function W that satisfies the B.C. of the system and $L[W]$ is continuous, can be expanded by a convergent series of eigenfunctions:

$$W = \sum_{r=1}^{\infty} C_r W_r$$

where $C_r = \int_D W m[W_r] \, dD$, and.

$$\int_D W_r m[W_r] \, dD = 1.$$

(Forced Vibration)

$$-EI\frac{\partial^4 y(x,t)}{\partial x^4} + f(x,t) = m\frac{\partial^2 y(x,t)}{\partial t^2}$$

Suppose the free vibration problem has eigenfunctions $Y_r(x)$ $(r=1,2,\cdots)$,
orthonormal
associated with eigenvalues ω_r^2.

Let $y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\, q_r(t)$

$$\Rightarrow \sum_{r=1}^{\infty} m Y_r(x)\,\ddot{q}_r(t) + \sum_{r=1}^{\infty} EI\frac{d^4 Y_r(x)}{dx^4}\, q_r(t) = f(x,t)$$

$$\Rightarrow \ddot{q}_s(t) + \omega_s^2 q_s(t) = Q_s(t) \qquad (s=1,2,\cdots)$$

where generalized force,

$$Q_s(t) = \int_0^L f(x,t)\, Y_s(x)\, dx$$

q_s is the generalized coordinator. ?

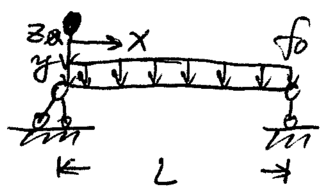
The generalized I.C. are.

$$\begin{cases} q_{s0} = \int_0^L m Y_s(x)\, y_0(x)\, dx \\ \dot{q}_{s0} = \int_0^L m Y_s(x)\, \dot{y}_0(x)\, dx \end{cases} \qquad (s=1,2,\cdots)$$

$$\therefore y(x,t) = \sum_{r=1}^{\infty} Y_r(x)\left[\frac{1}{\omega_r}\int_0^t Q_r(\tau)\sin\omega_r(t-\tau)\,d\tau + q_{r0}\cos\omega_r t + \frac{\dot{q}_{r0}}{\omega_r}\sin\omega_r t\right]$$

Continuous Vibration Problem I :

Uniform, simple supported beam, zero initial condition, uniform load.



m, EI

Differential equation for flexural vibration :

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 W}{\partial x^2}\right) + M\frac{\partial^2 W}{\partial t^2} = F(t)$$

$$\Rightarrow \quad EI\frac{\partial^4}{\partial x^4}W + m\frac{\partial^2}{\partial t^2}W = F(t) \quad (1)$$

I.C.&B.C. :

$$\begin{cases} W(x,0)=0; \ \frac{\partial}{\partial t}W(x,0)=0 \\ W(0,t)=0; \ \frac{\partial^2}{\partial x^2}W(0,t)=0 \\ W(L,t)=0; \ \frac{\partial^2}{\partial x^2}W(L,t)=0 \end{cases}$$

1° For free vibration (2)

$$EI\frac{\partial^4}{\partial x^4}W + m\frac{\partial^2}{\partial t^2}W = 0$$

B.C.

$$W(0,t)=0, \quad \frac{\partial^2}{\partial x^2}W(0,t)=0, \quad W(L,t)=0, \quad \frac{\partial^2}{\partial x^2}(L,t)=0$$

Suppose $W(x,t) = Y(x)q(t)$

$$\Rightarrow \quad \frac{EI}{Y(x)}Y^{(4)}(x) + \frac{m}{q(t)}q^{(2)}(t) = 0$$

x,t independent

$$\Rightarrow \quad \begin{cases} EIY^{(4)}(x) - w^2 m Y(x) = 0 \\ \ddot{q}(t) + w^2 q(t) = 0 \end{cases} \quad (3)$$

B.C.

$$Y(0)=0, \quad Y''(0)=0, \quad Y(L)=0, \quad Y''(L)=0$$

Let $L = EI\frac{d^4}{dx^4}$, $m=m$

$$\Rightarrow \quad L[Y] = w^2 m[Y] \quad (4)$$

Denote $\beta^4 = \frac{mw^2}{EI}$

$$\Rightarrow \quad Y^{(4)}(x) - \beta^4 Y(x) = 0$$

$$\Rightarrow \quad Y(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x$$

Using B.C.

$$\Rightarrow \quad \begin{cases} Y_r(x) = C_2\sin\beta_r x \\ \beta_r = \frac{r\pi}{L} \end{cases} \quad (r=1,2,\cdots)$$

normalize $Y_r(x)$:

$$\int_0^L mY_r^2(x)\,dx = 1$$

$$\Rightarrow \quad C_2 = \sqrt{\frac{2}{mL}}$$

∴ Eigen-value problem (4) have eigen gro[...]

$$\begin{cases} W_r^2 = \frac{EI}{m}\left(\frac{r\pi}{L}\right)^4 \\ Y_r(x) = \sqrt{\frac{2}{mL}}\sin\frac{r\pi x}{L} \end{cases} \quad (r=1,2,\cdots)$$

2° For original problem, write

$$W(x,t) = \sum_{r=1}^{\infty} Y_r(x)q_r(t)$$

$$\because \quad L[W] + m\left[\frac{\partial^2}{\partial t^2}W\right] = F(t)$$

$$\Rightarrow \quad \sum_{r=1}^{\infty} L[Y_r]q_r(t) + \sum_{r=1}^{\infty} m[Y_r]\ddot{q}_r(t) = F(t)$$

Integrate $\int_0^L Y_r[\ ]\,dx$

$$\Rightarrow \quad W_r^2 q_r(t) + \ddot{q}_r(t) = Q_r(t) \quad (5)$$

$$Q_r(t) = \int_0^L Y_r(x)F(t)\,dx = \sqrt{\frac{2}{mL}}\frac{1}{\beta_r}(1-\cos r\pi)f_0 u(t)$$

I.C.

$$q_{r0}=0, \quad \dot{q}_{r0}=0$$

$$q_r(t) = \int_0^t \frac{1}{\omega_r} \sin(\omega_r(t-\tau))\, Q_r(\tau)\, d\tau$$

$$= \int_0^t \frac{1}{\omega_r} \sin(\omega_r(t-\tau)) \sqrt{\frac{2}{mL}}\, \frac{2}{\beta_r} f_0\, u(\tau)\, d\tau \qquad (r=1,3,5,\cdots)$$

$$= \sqrt{\frac{2}{mL}}\, \frac{2f_0}{\beta_r \omega_r^2} (1-\cos\omega_r t)\, u(t)$$

$$\therefore W(x,t) = \sum_{r=1,3,5} \left[\frac{4 f_0 L^4}{EI (r\pi)^5} \sin\frac{r\pi x}{L} \cdot \left(1 - \cos\sqrt{\frac{EI}{m}}\frac{r^2\pi^2 t}{L^2}\right)\right] u(t)$$

$\square$.

Summary: 5 problems:

1° Original problem (IBC) (PDE)

2° Free vibration problem (B.C.) (PDE)

3° Variable separate problem (B.C.) (ODE)

4° Eigenvalue problem

5° Differential equation for generalized coordinate (IC) (ODE)