I

## The Delta method.

**Thm:** Let $\{\underline{Z}_n\}$ be a sequence of r.v. in $\mathbb{R}^d$, $\underline{b} \in \mathbb{R}^d$, and $a_n(\underline{Z}_n - \underline{b}) \xrightarrow{d} \underline{X}$, with $a_n \to \infty$.
Suppose $g: \mathbb{R}^d \to \mathbb{R}^r$, $\dot{g}$ continuous at $\underline{b}$,
then $a_n(g^T(\underline{Z}_n) - g^T(\underline{b})) \xrightarrow{d} \dot{g}^T(\underline{b})\underline{X}$.

Note: the delta method resembles $d\underline{g}(\underline{x}) = \nabla \underline{g}\, d\underline{x}$

proof:

$\because$ $a_n \| \underline{Z}_n - \underline{b} \| \to \|\underline{X}\|$

$\therefore$ $\| \underline{Z}_n - \underline{b} \| = \frac{1}{a_n}(a_n \|\underline{Z}_n - \underline{b}\|) \to 0$

$\therefore$ $\underline{Z}_n - \underline{b} \xrightarrow{p} 0$.

If $\underline{Y}_n$ is on line segment between $\underline{Z}_n$ and $\underline{b}$, then $\underline{Y}_n \xrightarrow{p} \underline{b}$ and $\dot{g}(\underline{Y}_n) \xrightarrow{p} \dot{g}(\underline{b})$. (continuous mapping thm)

Using Lagrange mid-value theorem, and Slutsky's thm

$$a_n(g^T(\underline{Z}_n) - g^T(\underline{b})) = a_n \dot{g}^T(\underline{Y}_n)(\underline{Z}_n - \underline{b}) \xrightarrow{d} \dot{g}^T(\underline{b})\underline{X}$$

$\square$.

E.g.: 1° $X \sim \mathcal{E}(\frac{1}{\theta})$, then $\sqrt{n}(\bar{X}_n - \theta) \xrightarrow{d} N(0, \theta^2)$

Then let $g(\theta) = P(X > x) = e^{-\frac{x}{\theta}}$,

we have $\sqrt{n}(g(\bar{X}_n) - g(\theta)) \xrightarrow{d} N(0, \dot{g}(\theta)^2 \theta^2)$

$\therefore \sqrt{n}\left(e^{-\frac{x}{\bar{X}_n}} - e^{-\frac{x}{\theta}}\right) \xrightarrow{d} N\left(0, \frac{x^2 e^{-\frac{2x}{\theta}}}{\theta^2}\right)$

2° $X \sim \text{Bernoulli}(n, p)$, then $\sqrt{n}(\bar{X}_n - p) \xrightarrow{d} N(0, pq)$

Let $g(p) = \log \frac{p}{1-p}$,

then $\sqrt{n}\left(\log \frac{\bar{X}_n}{1-\bar{X}_n} - \log \frac{p}{1-p}\right) \xrightarrow{d} N\left(0, \frac{1}{pq}\right)$

3° (Variance stablizing transformation)

$X \sim \text{Bernoulli}(n, p)$, find $g(p)$, s.t. $\dot{g}(p) \cdot pq = c^2$.

when $c = \frac{1}{2}$.

$\because g(p) =$

$\therefore g(p) = \arcsin(\sqrt{p})$

4° $\hat{\rho}$ sample correlation coefficient.

$$\hat{\rho} = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{X})}{\sqrt{\sum_{i=1}^{n} (X_i - \bar{X})^2 \cdot \sum_{i=1}^{n} (Y_i - \bar{Y})^2}}$$

$\sqrt{n}(\hat{\rho} - \rho) \xrightarrow{d} n(0, (1-\hat{\rho}^2)^2)$. Fisher's z-transform.