(2) Def: $X(u,t)$ is differentiable in the m.s.s. at $t=t_0$

$\exists$ r.p. $Y(u,t)$, $\forall \varepsilon>0$, $\exists\, \delta t>0$, s.t.

$$\mathbb{E}\left|\frac{X(u,t)-X(u,t_0)}{\delta t}-Y(u,t_0)\right|^2<\varepsilon,\quad \forall |t-t_0|<\delta t.$$

Thm: $X(u,t)$ is a second-order r.p. with $T=\mathbb{R}$,

$X(u,t)$ is differentiable in the m.s.s. at $t=t_0$,

$$\Longleftrightarrow \frac{\partial^2 R_X(t_1,t_2)}{\partial t_1 \partial t_2} \text{ exists at } (t_0,t_0).$$

In addition, if $X(u,t)$ is w.s.s., and $\frac{\partial^2 R_X(\tau)}{\partial \tau^2}$ exists at $\tau=0$, then $X(u,t)$ is differentiable for all $t\in\mathbb{R}$.

Note: 1° If $X(u,t)$ is differentiable $\forall t\in\mathbb{R}$, and $Y(u,t)=\frac{d}{dt}X(u,t)$, then expectation and differentiation are interchangeable:

$$\mathbb{E}\,Y(u,t) = \mathbb{E}\left[\frac{d}{dt}X(u,t)\right] = \frac{d}{dt}\mathbb{E}X(u,t) = \frac{d}{dt}m_X(t)$$

$$R_Y(t_1,t_2) = \mathbb{E}\left[\frac{d}{dt_1}X(u,t_1)\,\frac{d}{dt_2}X^*(u,t_2)\right] = \frac{\partial^2}{\partial t_1\partial t_2}R_X(t_1,t_2)$$

2° If $X(u,t)$ is w.s.s., then $Y(u,t)$ is also w.s.s.:

In addition,

$$m_Y(t)=0,\quad R_Y(t_1,t_2) = -\frac{d^2}{d\tau^2}R_X(t_1-t_2)$$

3° (spectral test for differentiability) If we see differentiation as:

$$\underset{\text{w.s.s.}}{X(u,t)} \longrightarrow \underset{\text{differentiator}}{\boxed{\text{LTI}\ \mathbb{D}}} \longrightarrow Y(u,t)$$

$$Y(u,t)=\mathbb{D}\,X(u,t)\equiv\frac{d}{dt}X(u,t)$$

Sys. fn. $D(f)=i2\pi f$.

Then $Y(u,t)$ exists in m.s.s. if it has finite power:

$$\int_{-\infty}^{+\infty} S_Y(f)\,df = \int_{-\infty}^{+\infty} S_X(f)\,|i2\pi f|^2\,df<\infty$$

4° (Ornstein-Uhlenbeck process) is a Gaussian process with $R_X(\tau) = Ae^{-B|\tau|}$. It is the only stationary Markov process.

$\because \frac{d^2}{d\tau^2} R_X(\tau)$ does not exist at $\tau = 0$,

$\therefore X(u,t)$ is not differentiable in m.s.s., $\forall t \in \mathbb{R}$.

E.g.: Assume $X(u,t)$ is arbitrarily differentiable in m.s.s., operation

$$\underset{\text{w.s.s.}}{X(u,t)} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H(f)\end{matrix}} \longrightarrow Y(u,t)$$

is defined by $$\sum_{i=0}^{I} a_i \frac{d^i}{dt^i} X(u,t) = \sum_{j=0}^{J} \frac{d^j}{dt^j} Y(u,t). \qquad (*)$$

Find 2^{nd} order description of $Y(u,t)$.

Approaches: 1° Frequency domain

(a) Find $H(f)$

(b) $S_Y(f) = S_X(f)\,|H(f)|^2$

(c) $R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(f)\}$

2° Time domain.

(a) Find cor. fn. of LHS $(*)$

(b) Find cor. fn. of RHS $(*)$

(c) Compare coe. to get $R_Y(\tau)$.

Approach 1° fails if either $X(u,t)$ is not w.s.s. or the LTI.

Approach 2° works for non-w.s.s. inputs and operators, but step (c) is difficult.