

(2) Def:  $X(u,t)$  is differentiable in the m.s.s. ~~at~~ ~~at~~ <sup>at  $t=t_0$</sup>   
 $\exists$  r.p.  $Y(u,t)$ ,  $\forall \epsilon > 0, \exists \delta > 0, \forall |t-t_0| < \delta$ , s.t.  

$$E \left| \frac{X(u,t) - X(u,t_0)}{\delta t} - Y(u,t_0) \right|^2 < \epsilon, \forall |t-t_0| < \delta$$

Thm:  $X(u,t)$  is a second-order r.p. with  $\mathcal{T} = \mathbb{R}$ ,  
 $\bullet$   $X(u,t)$  is differentiable in the m.s.s. at  $t=t_0$ ,  
 $\iff \frac{\partial^2 R_X(t_1, t_2)}{\partial t_1 \partial t_2}$  exists at  $(t_0, t_0)$ .

In addition, if  $X(u,t)$  is w.s.s., and  $\frac{\partial^2 R_X(t)}{\partial t^2}$  exists at  $t=0$ ,  
then  $X(u,t)$  is differentiable for all  $t \in \mathbb{R}$ .

Note: 1° If  $X(u,t)$  is differentiable  $\forall t \in \mathbb{R}$ , and  $Y(u,t) = \frac{d}{dt} X(u,t)$ ,  
then expectation and differentiation are interchangeable:

$$E Y(u,t) = E \left[ \frac{d}{dt} X(u,t) \right] = \frac{d}{dt} E X(u,t) = \frac{d}{dt} m_X(t)$$

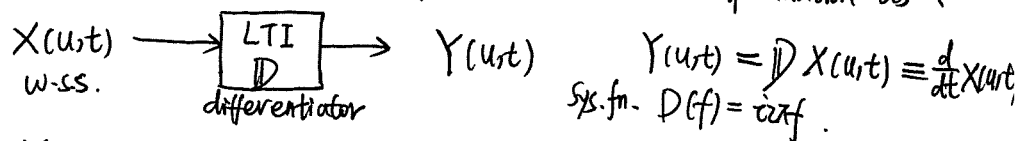
$$R_Y(t_1, t_2) = E \left[ \frac{d}{dt_1} X(u,t_1) \frac{d}{dt_2} X(u,t_2) \right] = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1, t_2)$$

2° If  $X(u,t)$  is w.s.s., then  $Y(u,t)$  is also w.s.s.:

In addition,

$$m_Y(t) = 0, \quad R_Y(t_1, t_2) = \frac{d^2}{dt^2} R_X(t_1 - t_2)$$

3° (spectral test for differentiability) If we see differentiation as:



Then  $Y(u,t)$  exists in m.s.s. if it has finite power:  
 $\int_{-\infty}^{\infty} S_Y(f) df = \int_{-\infty}^{\infty} S_X(f) |i\omega f|^2 df < \infty$

4° (Ornstein-Uhlenbeck process) is a Gaussian ~~process~~ process with  $R_X(\tau) = Ae^{-B|\tau|}$ . It is the only stationary Markov process -  
 $\therefore \frac{d^2}{d\tau^2} R_X(\tau)$  ~~does~~ not exist at  $\tau=0$ ,  
 $\therefore X(u,t)$  is not differentiable in m.s.s.,  $\forall t \in \mathbb{R}$ .

E.g.: Assume  $X(u,t)$  is arbitrarily differentiable in m.s.s., operation

$X(u,t)$  ~~w.s.s.~~  $\xrightarrow{\text{LTI H(f)}} Y(u,t)$

is defined by 
$$\sum_{i=0}^I a_i \frac{d^i}{dt^i} X(u,t) = \sum_{j=0}^J \frac{d^j}{dt^j} Y(u,t). \quad (*)$$

Find 2<sup>nd</sup> order ~~description~~ description of  $Y(u,t)$ .

Approaches = 1° Frequency domain

(a) Find  $H(f)$

(b)  $S_Y(f) = S_X(f) |H(f)|^2$

(c)  $R_Y(\tau) = \mathcal{F}^{-1}\{S_Y(f)\}$

2° Time domain.

(a) Find ~~cor. fn.~~ cor. fn. of LHS (\*)

(b) Find cor. fn. of RHS (\*)

(c) compare ~~coe.~~ coe. to get  $R_Y(\tau)$ .

Approach 1° fails if either  $X(u,t)$  is not w.s.s. or the LTI.

Approach 2° works for non-w.s.s. inputs and operators, but step (c) is difficult.