Distance to a subspace not attainable – an Example

The following comes from "Functional Analysis" by Angus Taylor. Let X be the subspace of $C([0,1],\mathbb{R})$ with the "sup" norm, consisting of all continuous functions x on $[0,1]$ satisfying $x(0)=0$. Then let X_0 be the closed subspace of all $x \in X$ such that $\int_0^1 x(t)dt = 0$.

<u>Claim</u> If $x_1 \in X$ with $\|x_1\| = 1$, then $d(x_1, X_0) < 1$ where "d" is the distance to the subspace X_0:

$$d(x_1, X_0) = \inf\{\|x_1 - x\|, x \in X_0\}.$$

Proof: It suffices to show that given $x_1 \in X$ with $\|x_1\| = 1$, then $\|x_1 - x\| < 1$ for some $x \in X_0$. Arguing negatively, suppose there is an $x_1 \in X$ with $\|x_1\| = 1$ but $\|x_1 - x\| \geq 1$ for all $x \in X_0$. For each $y \in X \setminus X_0$ let

$$c = \frac{\int_0^1 x_1(t)dt}{\int_0^1 y(t)dt}.$$

Then $x_1 - cy \in X_0$ and thus $1 \leq \|x_1 - (x_1 - cy)\| = |c|\|y\|$, or

$$\left|\int_0^1 y(t)dt\right| \leq \left|\int_0^1 x_1(t)dt\right| \|y\| \tag{1}$$

for each $y \in X$. We can make $\left|\int_0^1 y(t)dt\right|$ as close to one as we please while maintaining $\|y\| = 1$ (e.g. let $y_n(t) = t^{1/n}$ and let n be very large). But since $\|x_1\| = \max_{[0,1]}|x_1(t)| = 1$ and $x_1(0) = 0$, the continuity of x_1 implies that

$$\left|\int_0^1 x_1(t)dt\right| = \delta < 1,$$

so that (1) becomes

$$\left|\int_0^1 y_n(t)dt\right| \leq \delta < 1$$

a contradiction. QED