Dynamics of Systems of N particles-

Theorems: ① $\underline{r}_{cm} \equiv \frac{\sum_i m_i \underline{r}_i}{m}$ $\quad \underline{P} \equiv \sum_i \underline{P}_i$

then $\underline{P} = m\underline{v}_{cm}$ $\quad \left(\sum_{i=1}^{n} \underline{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{F}_{ij} = \sum_{i=1}^{n} \dot{\underline{P}}_i \right.$

Equation of motion of CM: $\quad \left. \Rightarrow \sum_i \underline{F}_i = \dot{\underline{p}} = m\dot{\underline{v}}_{cm} = m\underline{a}_{cm} \right)$

$\sum_i \underline{F}_i = \dot{\underline{p}} = m\,\underline{a}_{cm}$, where $\underline{F}_i$ is the external force on i

② $\underline{L} \equiv \sum_{i=1}^{n} (\underline{r}_i \times m_i \underline{v}_i)$, $\underline{N} \equiv \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i$ ← (external force)

(first form of torque equation) $\left(\frac{d\underline{L}}{dt}\right)_{fixed} = \sum_{i=1}^{n} (\underline{r}_i \times m_i \underline{a}_i)$

$= \sum_{i=1}^{n} \left(\underline{r}_i \times \left(\underline{F}_i + \sum_{j=1}^{n} \underline{F}_{ij}\right)\right)$

$= \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i + \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{r}_i \times \underline{F}_{ij}$ $\quad (\underline{r}_i \times \underline{F}_{ij} + \underline{r}_j \times \underline{F}_{ji} =$

$= \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i = \underline{N}$

Hence, $\left(\frac{d\underline{L}}{dt}\right)_{fixed} = \underline{N}$

③ $T \equiv \sum_i \frac{1}{2} m_i v_i^2$

$\therefore T = \sum_i \frac{1}{2} m_i (\underline{v}_i \cdot \underline{v}_i)$

$= \sum_i \frac{1}{2} m_i (\underline{v}_{cm} + \bar{\underline{v}}_i) \cdot (\underline{v}_{cm} + \bar{\underline{v}}_i)$

$= \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i \bar{v}_i^2 + \underline{v}_{cm} \cdot \sum_i m_i \bar{\underline{v}}_i$ $\quad (\sum_i m_i \bar{\underline{v}}_i = 0)$

$= \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i \bar{v}_i^2$

Hence, $T = \frac{1}{2} m v_{cm}^2 + \sum_i \frac{1}{2} m_i \bar{v}_i^2$

④ $\underline{r}_i = \underline{r}_{cm} + \bar{\underline{r}}_i$, $\underline{v}_i = \underline{v}_{cm} + \bar{\underline{v}}_i$

$\therefore \underline{L} = \sum_i (\underline{r}_{cm} + \bar{\underline{r}}_i) \times m_i (\underline{v}_{cm} + \bar{\underline{v}}_i)$

$= \underline{r}_{cm} \times m\underline{v}_{cm} + \underline{r}_{cm} \times \sum_i m_i \bar{\underline{v}}_i + \left(\sum_i m_i \bar{\underline{r}}_i\right) \times \underline{v}_{cm} +$
$\sum_i (\bar{\underline{r}}_i \times m_i \bar{\underline{v}}_i)$

$= \underline{r}_{cm} \times m\underline{v}_{cm} + \sum_i (\bar{\underline{r}}_i \times m_i \bar{\underline{v}}_i)$

Hence, $\underline{L} = \underline{r}_{cm} \times m\underline{v}_{cm} + \sum_i (\bar{\underline{r}}_i \times m_i \bar{\underline{v}}_i)$

$= \underline{r}_{cm} \times m\underline{v}_{cm} + \bar{\underline{L}}$ $\quad (\bar{\underline{L}} \equiv \sum_i \bar{\underline{r}}_i \times m_i \bar{\underline{v}}_i)$

note : (second form of torque equations)

$\because \left(\frac{d\underline{L}}{dt}\right)_{\text{fixed}} = \underline{N}$, $\underline{L} = \underline{r}_{cm} \times m\underline{v}_{cm} + \underline{\bar{L}}$ $\left(\underline{\bar{L}} \equiv \sum_i \underline{\bar{r}_i} \times m\underline{\bar{v}_i} \right)$

$\therefore \underline{\dot{r}}_{cm} \times m\underline{v}_{cm} + \underline{r}_{cm} \times m\underline{a}_{cm} + \left(\frac{d\underline{\bar{L}}}{dt}\right)_{\text{fixed}} = \underline{N}$.

$\therefore \underline{r}_{cm} \times \left(\sum_i \underline{F}_i\right) + \left(\frac{d\underline{\bar{L}}}{dt}\right)_{\text{fixed}} = \sum_i \underline{r}_i \times \underline{F}_i$

$\therefore \left(\frac{d\underline{\bar{L}}}{dt}\right)_{\text{fixed}} = \sum_i (\underline{r}_i - \underline{r}_{cm}) \times \underline{F}_i = \sum_i \underline{\bar{r}_i} \times \underline{F}_i$

Hence, $\left(\frac{d\underline{\bar{L}}}{dt}\right)_{\text{fixed}} = \underline{\bar{N}}$ $\left(\underline{\bar{N}} \equiv \sum_i \underline{\bar{r}_i} \times \underline{F}_i \right)$ (external).