CE507 Energy method

1. Strain energy density due to normal stress:

$$W_0 = \frac{\delta W}{\delta V} = \int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x = \frac{1}{2}\sigma_x \varepsilon_x$$

Strain energy density due to shear stress:

$$W_0 = \frac{\delta W}{\delta V} = \frac{1}{2}(\sigma_{xy}\varepsilon_{xy} + \sigma_{yx}\varepsilon_{yx})$$

general strain energy density for linear problems:

$$W_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}(\lambda \varepsilon_{kk}^2 + 2\mu \varepsilon_{ij}\varepsilon_{ij})$$

$$= \frac{1}{4\mu}\sigma_{ij}\sigma_{ij} - \frac{\lambda}{4\mu(2\mu+3\lambda)}\sigma_{kk}^2$$

$$= \frac{1}{2}\left(\frac{1+\nu}{E}\sigma_{ij}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}^2\right)$$

$$\frac{\partial W_0}{\partial \varepsilon_{ij}} = \begin{cases} \lambda \varepsilon_{kk} + 2\mu \varepsilon_{ij}, & i=j \\ 2\mu \varepsilon_{ij}, & i \neq j \end{cases}$$

$$= \sigma_{ij}$$

2. virtual strain energy density

$$\delta W_0 = \int \sigma_{ij} \, d(\delta \varepsilon_{ij})$$

virtual strain energy

$$\delta W_V = \int_V \delta W_0 \, dV$$

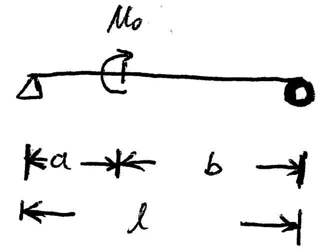
total virtual work (it can be proven to be)

$$\delta W_E = \iiint_V T_{ij} \, \delta u_{i,j} \, dV$$

The principle of virtual displacement: (Π: potential energy)

$$\cancel{\delta W_V = \delta W_E} \quad \delta \Pi = \delta W_V - \delta W_E = 0$$

3. Simply supported beam with external moment



$\because$ B.C. are
$$\begin{cases} v(0) = v(l) = 0 \\ v''(0) = v''(l) = 0 \end{cases}$$

$\therefore$ Suppose $v(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}$

Then $\delta v = \sum_{n=1}^{\infty} \delta A_n \sin \frac{n\pi x}{l}$

Total strain energy is

$$\begin{aligned} W_v &= \iiint W_0 \, dv \\ &= \iiint \frac{\sigma_x^2}{2E} \, dv \leftarrow \text{Why ignoring other terms in } W_0\text{?} \quad \left(W_0 = \frac{1}{2}\left(\frac{1+\nu}{E} \sigma_{ij}\sigma_{ij} - \frac{\nu}{E}\Theta^2 \right) \right) \\ &= \iiint \frac{E}{2} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 y^2 \, dx\,dy\,dz \\ &= \int_0^l \frac{EI}{2} \left(\frac{\partial^2 v}{\partial x^2} \right)^2 dx \\ &= \int_0^l \frac{EI}{2} \left[\sum_{n=1}^{\infty} -A_n \left(\frac{n\pi}{l} \right)^2 \sin \frac{n\pi x}{l} \right]^2 dx \\ &= \frac{EI}{2} \sum_{n=1}^{\infty} \int_0^l A_n^2 \left(\frac{n\pi}{l} \right)^4 \sin^2 \frac{n\pi x}{l} \, dx \\ &= \frac{\pi^4 EI}{4 l^3} \sum_{n=1}^{\infty} A_n^2 n^4 \end{aligned}$$

$\therefore$ Virtual strain energy is $\delta W_v = \frac{\pi^4 EI}{4 l^3} \sum_{n=1}^{\infty} 2 A_n \delta A_n \cdot n^4$

The virtual work is

$$\delta W_E = M \, \delta \left(\frac{dv}{dx} \Big|_{x=a} \right) = M \sum_{n=1}^{\infty} \delta A_n \cdot \frac{n\pi}{l} \cdot \cos \frac{n\pi a}{l}$$

From $\delta \Pi = \delta W_V - \delta W_E = 0$,

$$\sum_{n=1}^{\infty} \left[\frac{\pi^4 EI}{4l^3} \cdot 2A_n \cdot n^4 - M \frac{n\pi}{l} \cdot \cos\frac{n\pi a}{l} \right] \delta A_n = 0$$

Since δA_n $(n=1,2,\cdots)$ is arbitary,

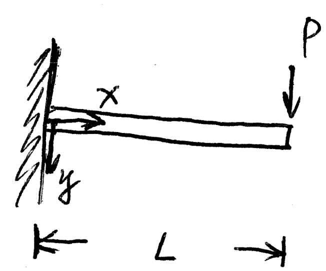
$$A_n = \frac{2l^2 M}{n^3 \pi^3 EI} \cos\frac{n\pi a}{l} \qquad (n=1,2,\cdots)$$

$$\therefore \quad V(x) = \frac{2l^2 M}{\pi^3 EI} \sum_{n=1}^{\infty} \frac{1}{n^3} \cos\frac{n\pi a}{L} \sin\frac{n\pi x}{l}$$

4. Cantilever beam loaded by an end force

B.C. are

$$\begin{cases} V(0) = V'(0) = 0 \\ V''(0) = \frac{PL}{EI} \\ V''(L) = 0 \end{cases}$$



Suppose $V(x) = \sum_{n=1,3,5,\cdots} A_n \left(1 - \cos\frac{n\pi x}{2L}\right)$

Total strain energy is

$$\begin{aligned} W_V &= \frac{EI}{2} \int_0^L \left(\frac{\partial^2 V}{\partial x^2}\right)^2 dx \\ &= \frac{EI}{2} \int_0^L \left[\sum_{n=1,3,5,\cdots} A_n \left(\frac{n\pi}{2L}\right)^2 \cos\frac{n\pi x}{2L} \right]^2 dx \\ &= \frac{EI}{2} \sum_{n=1,3,5,\cdots} \int_0^L A_n^2 \left(\frac{n\pi}{2L}\right)^4 \cos^2\frac{n\pi x}{2L}\, dx \\ &= \frac{\pi^4 EI}{64 L^3} \sum_{n=1,3,5,\cdots} A_n^2 \cdot n^4. \end{aligned}$$

$\therefore$ Virtual strain energy is

$$\delta W_V = \frac{\pi^4 EI}{64 L^3} \sum_{n=1,3,5,\cdots} 2A_n \delta A_n n^4$$

The virtual work is

$$\delta W_E = P\,\delta V|_{x=L} = P \sum_{n=1,3,5,\cdots} \delta A_n$$

From $\delta \Pi = \delta W_v - \delta W_E = 0$,

$$\sum_{n=1,3,5,\cdots} \left(\frac{\pi^4 EI}{64L^3} \cdot 2A_n n^4 - P \right) \delta A_n = 0$$

$$\therefore \quad A_n = \frac{32PL^3}{n^4 \pi^4 EI} \qquad (n=1,3,5,\cdots)$$

Note: 1° The series form of displacement function is determined by boundary conditions, and doesn't change with ~~so~~ different load forces. Hence the form of total strain energy.

5. bending ~~eq~~ equation:

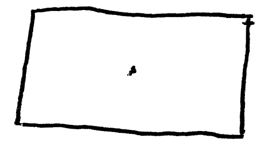
$$EIv^{(4)} = q$$

For general distribution load q,

$$\Pi = \int_0^L \frac{EI}{2} \left(\frac{\partial^2 v}{\partial x^2}\right)^2 - qv\,dx$$

Rayleigh – Ritz method

1. Torsion of a rectangular shaft:



Strain energy density is:

$$W_0 = \frac{1}{2}\sigma_{ij}\,\varepsilon_{ij}$$

$$= \tau_{zx}\varepsilon_{zx} + \tau_{zy}\varepsilon_{zy}$$

$$= \frac{1}{2\mu}\left(\tau_{zx}^2+\tau_{zy}^2\right)$$

$$= \frac{1}{2\mu}\left[\left(\frac{\partial\phi}{\partial y}\right)^2+\left(-\frac{\partial\phi}{\partial x}\right)^2\right]$$

Strain energy per unit length is:

$$W_V = \frac{1}{2\mu}\iint \left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial x}\right)^2 dx\,dy$$

The work done by external force is:

$$W_E = M\alpha = 2\alpha\iint \phi\, dx\,dy$$

$$\therefore\ \Pi = W_V - W_E = \iint \frac{1}{2\mu}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right] - 2\alpha\phi\ dx\,dy$$

Suppose $\phi = (x^2-a^2)(y^2-b^2)\sum_{m,n=0,2,4,\cdots} C_{mn}x^m y^n$

Take a 1-term formula, $\phi = C_{00}(x^2-a^2)(y^2-b^2)$,

then

$$\Pi = \int_{-b}^{b}\int_{-a}^{a} \frac{4C_{00}^2}{2\mu}\left[x^2(y^2-b^2)^2+y^2(x^2-a^2)^2\right] - 2\alpha C_{00}(x^2-a^2)(y^2-b^2)\,dx\,dy$$

$$= 8\int_0^b\int_0^a \frac{C_{00}^2}{\mu}\left[x^2(y^2-b^2)^2+y^2(x^2-a^2)^2\right] - \alpha C_{00}(x^2-a^2)(y^2-b^2)\,dx\,dy$$

$$= \frac{32}{45}a^3b^3\left[\frac{2C_{00}^2}{\mu}(a^2+b^2) - 5\alpha C_{00}\right]$$

By Rayleigh-Ritz method, $\frac{\partial \Pi}{\partial C_{00}} = 0$,

$$\therefore \quad C_{00} = \frac{5\mu\alpha}{4(a^2+b^2)}$$

$\therefore$ The 1-term formula is $\phi = \frac{5\mu\alpha}{4(a^2+b^2)}(x^2-a^2)(y^2-b^2)$