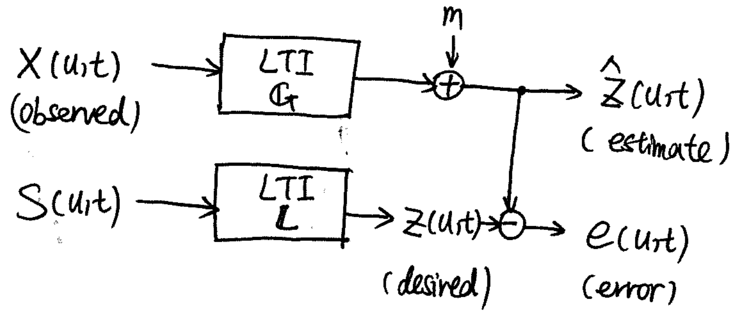
[Estimation]

(Chap 15-6)

AMMSE estimator for w.s.s. random sequences as LTI operators

$X(u,t)$, $S(u,t)$ are jointly w.s.s., with known 2^{nd} moment description.

(1) Minimize $MSE = \mathbb{E}|e(u,t)|^2 = R_e(0) = R_{e_0}(0) + |m_e|^2$.



We have
$$\begin{cases} \hat{Z} = m + GX \\ Z = LS \\ e = \hat{Z} - Z \end{cases}$$

1° $m_e = m_{\hat{Z}} - m_Z = m + m_X G(0) - m_S L(0)$

Let $m_e = 0$, we have $m = m_S L(0) - m_X G(0)$

2° $R_{e_0}(\tau) = \mathbb{E}\left\{ \left(\hat{Z}_0(u,t+\tau) - Z_0(u,t+\tau)\right)\left(\hat{Z}_0(u,t) - Z_0(u,t)\right)^* \right\}$

$= R_{\hat{Z}_0}(\tau) + R_{Z_0}(\tau) - R_{\hat{Z}_0 Z_0^*}(\tau) - R_{Z_0 \hat{Z}_0^*}(\tau)$

$\therefore S_{e_0}(f) = S_{\hat{Z}_0}(f) + S_{Z_0}(f) - S_{\hat{Z}_0 Z_0^*}(f) - S_{Z_0 \hat{Z}_0^*}(f)$

$= S_{X_0}(f)|G(f)|^2 + S_{S_0}(f)|L(f)|^2 - G(f) S_{X_0 S_0^*}(f) L^*(f)$

$- L(f) S_{S_0 X_0^*}(f)\, G^*(f)$

$\therefore R_{e_0}(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{e_0}(f)\, df$

$= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{X_0}(f) \left| G(f) - \frac{S_{S X_0^*}(f) L(f)}{S_{X_0}(f)} \right|^2 df +$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} S_{s_0}(f)\,|L(f)|^2 - \frac{|S_{s_0x_0^*}(f)\,L(f)|^2}{S_{x_0}(f)}\,df$$

The 2^{nd} term doesn't depend on $G(f)$, minimize $R_{e_0}(0)$ gives

$$G(f) = L(f)\,S_{s_0x_0^*}(f)\,(S_{x_0}(f))^{-1}.$$

Hence

$$\begin{cases} G_{opt}(f) = L(f)\,S_{s_0x_0^*}(f)\,(S_{x_0}(f))^{-1} \\ m_{opt} = m_s L(0) - m_x G_{opt}(0) \\ MMSE = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{s_0}(f)\,|L(f)|^2 - \frac{|S_{s_0x_0^*}(f)\,L(f)|^2}{S_{x_0}(f)}\,df \end{cases}$$

Note: 1° The optimal choices are called Wiener filters.

2° LMMSE estimator has a similar form, except it's noncentral.

(2) Minimize MSE with causal $\mathbb{G}$.

1° Step is the same.

2° $$R_{e_0}(0) = MMSE + \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{x_0}(f)\left|G(f) - \frac{L(f)\,S_{s_0x_0^*}(f)}{S_{x_0}(f)}\right|^2 df$$

Find <u>causal system $\mathbb{H}$</u> that simulates X, s.t. <u>$S_{x_0}(f) = |H(f)|^2$</u>,

then
$$R_{e_0}(0) - MMSE = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| H(f)G(f) - \frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right|^2 df$$

$$\text{(Parseval's thm)} = \sum_{\tau=-\infty}^{+\infty} \left| \mathcal{F}^{-1}\{H(f)G(f)\} - \mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right|^2$$

$$= \sum_{\tau=0}^{+\infty} \left| \mathcal{F}^{-1}\{H(f)G(f)\} - \mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right|^2 + \sum_{\tau=-\infty}^{-1} \left| \mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right|^2$$

$$1^{st}\ \text{term} = \sum_{\tau=-\infty}^{+\infty} \left| \mathcal{F}^{-1}\{H(f)G(f)\} - U_r(\tau)\,\mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right|^2$$

($U_r(\tau)$ is unit-step fn)

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| H(f)G(f) - \mathcal{F}\left\{U_r(\tau)\,\mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right\}\right|^2 df$$

$R_{e_0}(0)$ (Parseval's thm)

Minimize gives,

$$G(f) = H^{-1}(f)\,\mathcal{F}\left\{U_r(\tau)\,\mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right\}$$

Hence,

~~$m_{opt,causal}$~~

$$G_{opt,causal}(f) = \left(H(f)\right)^{-1}\mathcal{F}\left\{U_r(\tau)\,\mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right\}$$

$$m_{opt,causal} = m_s L(0) - m_x G_{opt,causal}(0)$$

$$\mathrm{MMSE}_{causal} = \cancel{\int_{-\frac{1}{2}}^{\frac{1}{2}} S_{s_0}(f)\,|L(f)|^2\, df}$$

$$\boxed{\begin{aligned}\mathrm{MMSE}_{causal} &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{s_0}(f)\,|L(f)|^2 df - \sum_{\tau=0}^{+\infty}\left|\mathcal{F}^{-1}\left\{\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}\right\}\right|^2 \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_{s_0}(f)\,|L(f)|^2 - \left|\mathbb{C}\left\{\frac{S_{s_0x_0^*}(f)\,L(f)}{H^*(f)}\right\}\right|^2 df\end{aligned}}$$

Note: Eliminating $\mathcal{F}\{U_r(\tau)\,\mathcal{F}^{-1}\{\cdot\}\}$ gives the MMSE estimator.

(Causal part operator $\mathbb{C}$ for $\mathcal{T} = \mathbb{Z}$)

we call $\mathbb{C} = \mathcal{F}\,U_r(\tau)\,\mathcal{F}^{-1}$ a <u>causal part operator</u>.

It's a linear operator.

suppose $\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)}$ is rational fn of z, and has no repeated poles.

we can write

$$\frac{L(f)\,S_{s_0x_0^*}(f)}{H^*(f)} = \sum_{n=1}^{D} \frac{C_n z^{k_n}}{z-d_n}$$

If $\begin{cases}|d_n|>1\\ k_n\geq 1\end{cases}$:

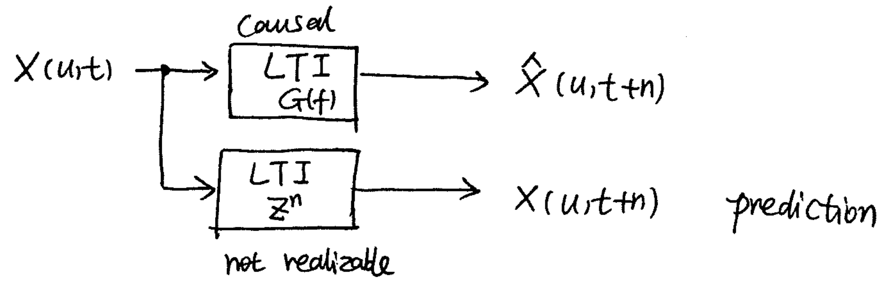
$$\frac{C_n z^{k_n}}{z-d_n} = \frac{(-d_n)^{-1} C_n z^{k_n}}{1-d_n^{-1}z} = C_n(-d_n)^{-1}\sum_{j=0}^{\infty} d_n^{-j} z^{j+k_n}.$$

then $\mathbb{C}\,\frac{C_n z^{k_n}}{z-d_n} = 0$

If $\begin{cases}|d_n|<1\\ k_n\leq 1\end{cases}$:

$$\frac{C_n z^{k_n}}{z-d_n} = \frac{C_n z^{k_n-1}}{1-d_n z^{-1}} = C_n\sum_{j=0}^{\infty} d_n^{j} z^{k_n-j-1}$$

E.g. : (pure prediction)



$m_X = 0$, $S_X(f) = \dfrac{1}{|1-az^{-1}|^2}$ $(|a|<1)$

The causal MMSE estimator,

with $S_{S_0X_0^*}(f) = S_X(f) = \dfrac{1}{|1-az^{-1}|^2}$

$$H(f) = \frac{1}{1-az^{-1}}, \quad L(f) = z^n$$

is
$$
\begin{aligned}
G_{opt,causal} &= (H(f))^{-1} \mathcal{C} \frac{L(f) S_{S_0X_0^*}(f)}{H^*(f)} \\
&= (H(f))^{-1} \mathcal{F} U_r(\tau) \mathcal{F}^{-1}\left\{ \frac{z^n}{1-az^{-1}} \right\} \\
&= (H(f))^{-1} \mathcal{F} U_r(\tau) \mathcal{F}^{-1}\left\{ \sum_{m=0}^{\infty} a^m z^{-m+n} \right\} \\
&= (H(f))^{-1} \mathcal{F} U_r(\tau) \left\{ a^{\tau+n} U_r(\tau+n) \right\} \\
&= (H(f))^{-1} \mathcal{F} \left\{ a^{\tau+n} U_r(\tau) \right\} \\
&= (H(f))^{-1} \frac{a^n}{1-az^{-1}} \\
&= a^n
\end{aligned}
$$

$\therefore$ $g_{opt,causal}(t) = \mathcal{F}^{-1} G_{opt,causal} = a^n \delta(t)$

$\therefore$ $\hat{X}(u,t+n) = (g * X)(u,t) = a^n X(u,t)$

is the estimated prediction.

AMMSE estimator for w.s.s. random waveform as LTI operators

Results are similar to AMMSE estimator for w.s.s. sequences.

(1) AMMSE estimator

$$G_{opt}(f) = \frac{L(f)\, S_{s_0 x_0^*}(f)}{S_{x_0}(f)}$$

$$m_{opt} = m_s L(0) - m_x G_{opt}(0)$$

$$MMSE = \int_{-\infty}^{+\infty} |L(f)|^2 \cdot \left[S_s(f) - \frac{|S_{s_0 x_0^*}(f)|^2}{S_{x_0}(f)} \right] df$$

$$\boxed{= \int_{-\infty}^{+\infty} |L(f)|^2 S_{s_0}(f) - S_{\hat{z}_0}(f)\, df}$$

(2) Causal AMMSE estimator

$$G_{opt,\,causal}(f) = \frac{1}{H(f)}\, \mathcal{F}\, U_r(t)\, \mathcal{F}^{-1}\left\{ \frac{L(f)\, S_{s_0 x_0^*}(f)}{H^*(f)} \right\}$$

where $S_{x_0}(f) = |H(f)|^2$, H is causal.

$$m_{opt,\,causal} = m_s L(0) - m_x G_{opt,\,causal}(0)$$

$$MMSE_{causal} = \int_{-\infty}^{+\infty} |L(f)|^2 S_s(f)\, df - \int_{0}^{+\infty} \left| \mathcal{F}^{-1}\left\{ \frac{L(f)\, S_{s_0 x_0^*}(f)}{H^*(f)} \right\} \right|^2 dt$$

$$= \int_{-\infty}^{+\infty} |L(f)|^2 S_s(f) - \left| \mathbb{C}\left\{ \frac{L(f)\, S_{s_0 x_0^*}(f)}{H^*(f)} \right\} \right|^2 df$$

$$\boxed{= \int_{-\infty}^{+\infty} |L(f)|^2 S_{s_0}(f) - S_{\hat{z}_0}(f)\, df}$$