Electrodynamics and the Maxwell's equations

Summary: Electrostatics and Magnetostatics

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 & \text{Gauss' law} \\ \nabla \times \mathbf{E} = 0 & \text{No name} \\ \nabla \cdot \mathbf{B} = 0 & \text{No name} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} & \text{Ampere's law} \end{cases}$$

These equations specify the divergence and curl of **E** and **B**, and therefore determine the field, given the charge and current densities.

Inside matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = 0 \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \mathbf{J}_f \end{cases}$$

where

$$\begin{cases} \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \\ \mathbf{H} = \dfrac{1}{\mu_0} \mathbf{B} - \mathbf{M} \end{cases}$$

For the set of equations to be closed, one has to supply the relation between **D**, **E** and **H**, **M**, which are called the constitutive relations.
For instance, in linear media,

$$\begin{cases} \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{H} = \dfrac{1}{\mu} \mathbf{B} \end{cases}$$

The force a charge q moving with velocity **v** experiences in a region of E field and B field is given by the Lorentz force law:

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

In the static cases, the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

is automatically satisfied. Since

$$\frac{\partial \rho}{\partial t} = 0$$

and

$$\nabla \cdot \mathbf{J} = \frac{1}{\mu_0} \nabla \cdot (\nabla \times \mathbf{B}) = 0$$

However, in electrodynamics, it is found that the two curl equations have to be modified.

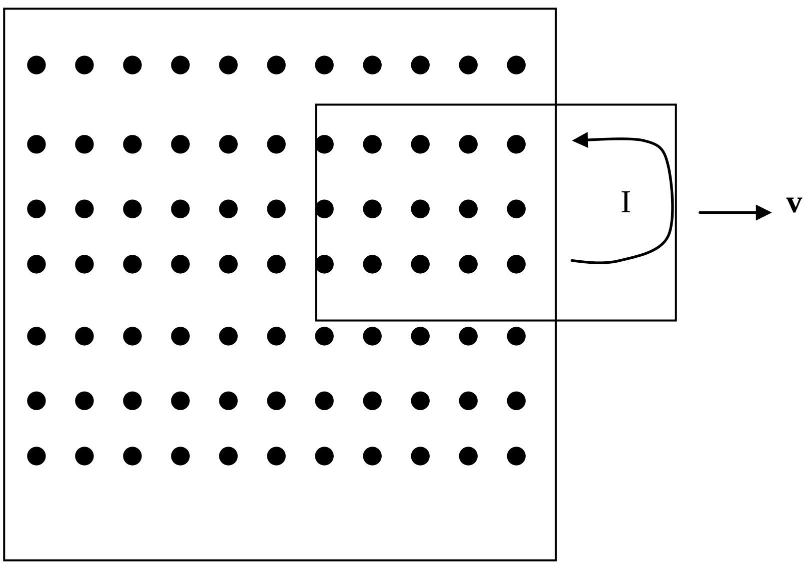
Electromagnetic induction

Faraday's Experiments

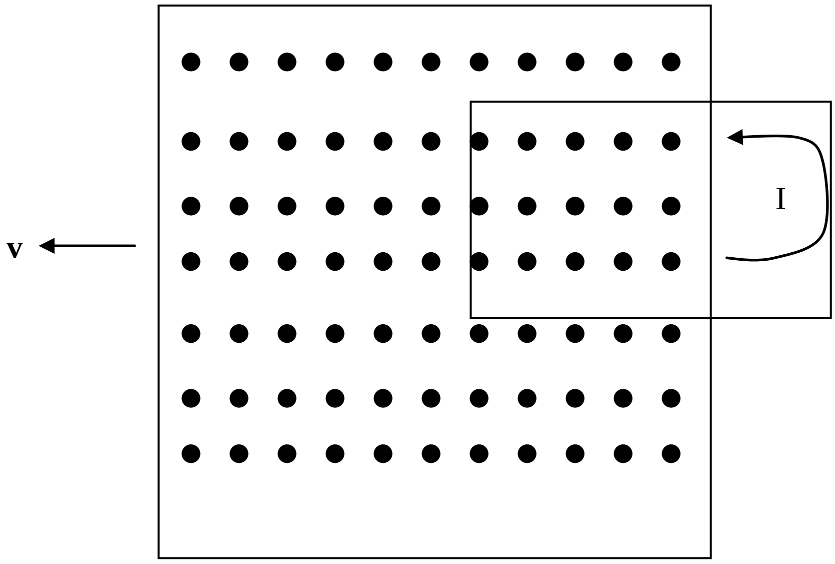
In the nineteenth century, Faraday performed a series of experiments which showed that in general, the electric field is not curl-free.

Experiment 1:

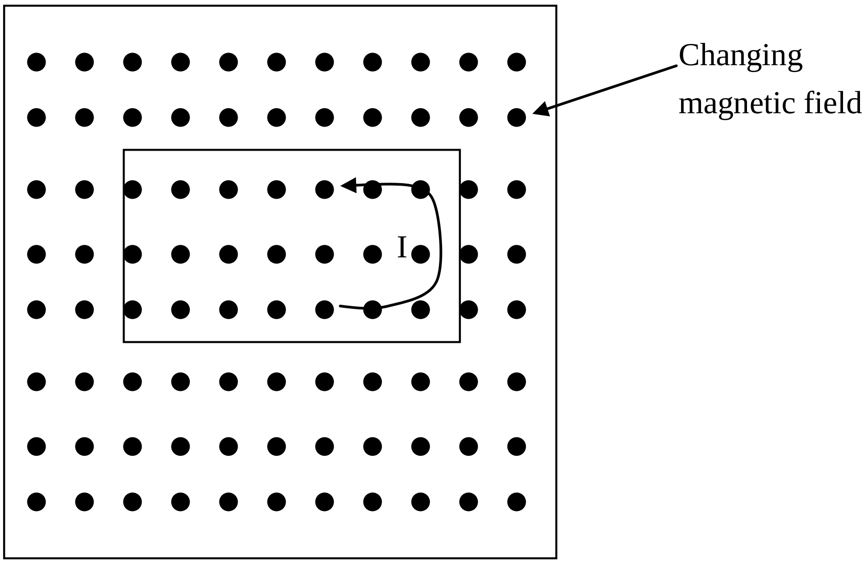
A loop of wire partly inside a magnetic field (assume uniform for simplicity) moving with velocity **v** perpendicular to the field.



Experiment 2:
A magnetic field partly inside a loop of wire moving to the opposite direction.



Experiment 3:
A loop at rest inside a changing magnetic field.



Observation: In all the experiments, there will be a current flowing.

Why?

There is a current because there is a force driving the charges to move. Let **f** be the force per unit charge. The electromotive force (emf) $\mathcal{E}$ is defined by

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l}$$

over a closed loop.

When there is a driving force, it is a "rule of thumb" that a current will be generated which is proportional to **f**:

$$\mathbf{J} = \sigma \mathbf{f} \ .$$

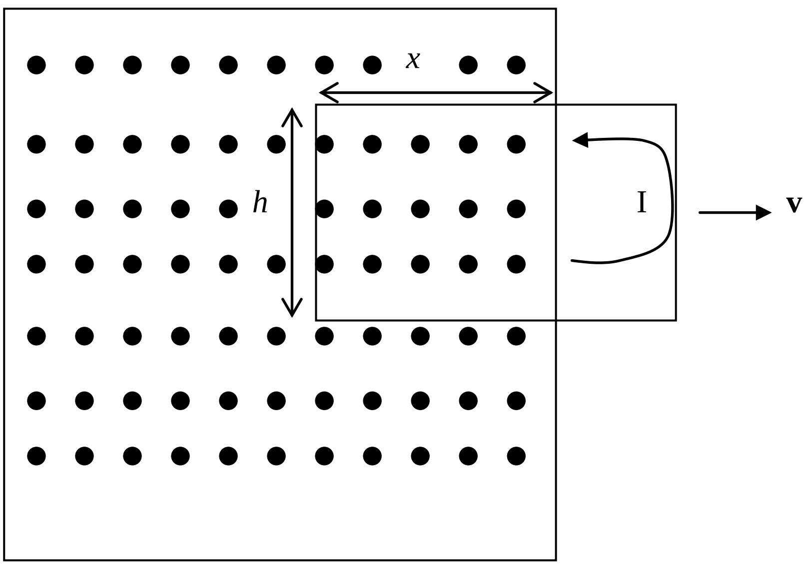
The proportionality constant σ is called the conductivity of the material, and its reciprocal

$$\rho = \frac{1}{\sigma}$$

is called the resistivity.

The source of this driving force in the Faraday's experiments has different interpretations though.

Experiment 1:
The force is due to the Lorentz force of charges in motion → Motional emf.



When the loop moves, the charges inside experience a force

$$\mathbf{f} = \mathbf{v} \times \mathbf{B}$$

which points upward with magnitude vB. Obviously only the left side of the loop contributes to the emf

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} \qquad (\text{counterclockwise as positive}).$$

Hence

$$\mathcal{E} = vBh.$$

Notice that the emf in this case can be related to the magnetic flux

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a} \qquad (\text{inward as positive})$$

through the loop.

Note that the sign convention of emf and flux has to be consistent by right hand rule.

In this particular case, obviously $\Phi = Bhx$
where x is the portion of the length of the loop inside the field.
Hence

$$\frac{d\Phi}{dt} = Bh\frac{dx}{dt} = -vBh$$

The relation is hence

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

which is called the flux rule.

It can be shown that:
The flux rule is valid in general for an arbitrary loop moving in a non-uniform B field.

Note: In experiment 1, there is no electric field, and we will not call this electromagnetic induction.

What about experiments 2 and 3?

Experiment 2, 3:
Imagine an observer in experiment 1 moving with velocity **v**. What he will observe is exactly that in experiment 2, viz., a loop at rest with a magnetic field moving to the right.
A current and hence electromotive force will still be observed. However,

this time there should be no Lorentz force due to magnetic field since the loop is not moving. Hence it can be concluded that there is an electric field.

Faraday's law:

Faraday proposed that a changing magnetic field will induce an electric field.

And the flux rule is still correct. However, this time the driving force is due to an induced electric field. Hence

$$\frac{d\Phi}{dt} = -\mathcal{E}$$

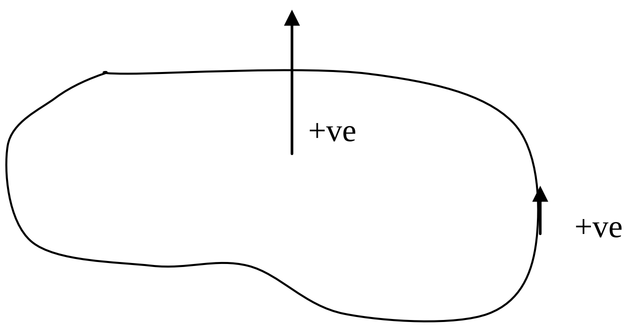
$$\boxed{\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\oint \mathbf{E}\cdot d\mathbf{l}} \qquad \text{(Faraday's law in integral form)}$$

$$\Rightarrow \int \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{a} = -\int (\nabla\times\mathbf{E})\cdot d\mathbf{a}$$

$$\Leftrightarrow \boxed{\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \qquad \text{(Faraday's law in differential form)}$$

Note that the minus sign denotes what is called the Lenz's law:

Nature abhors a change in flux.

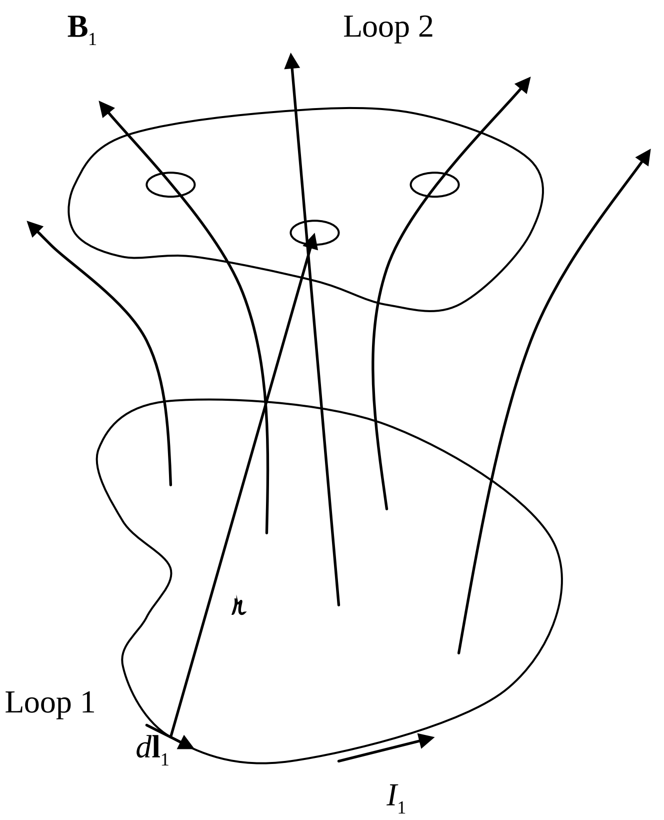


e.g., Flux increases → negative $\nabla\times\mathbf{E}$ → negative current → produces negative flux → opposes the change in flux

The induced electric field forms close loops (cf. $\nabla\times\mathbf{E}=0$ in electrostatics) and is divergence free. Therefore, the total electric field due to charges and changing magnetic field satisfies

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0, \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Inductance



Consider two wire loops 1 and 2. A current I_1 flowing in loop 1 will produce a magnetic field

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \frac{d\mathbf{l}_1 \times \hat{\imath}}{\imath^2}$$

The flux due to $\mathbf{B}_1$ through loop 2 is

$$\Phi_2 = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{a}_2$$

which is obviously proportional to I_1.

Define $$\Phi_2 = M_{21} I_1$$

where M_{21} is called the mutual inductance.

To determine the mutual inductance, notice that

$$\Phi_2 = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int_{S_2} \left(\nabla \times \mathbf{A_1}\right) \cdot d\mathbf{a}_2 = \oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l}_2$$

But we have derived that $\mathbf{A}_1 = \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \frac{d\mathbf{l}_1}{\mathscr{r}}$.

Hence

$$\Phi_2 = \oint_{C_2} \frac{\mu_0}{4\pi} I_1 \oint_{C_1} \frac{d\mathbf{l}_1}{\mathscr{r}} \cdot d\mathbf{l}_2 = I_1 \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

$$\Rightarrow M_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

which is called the Neumann formula.

The double integral is usually difficult to evaluate and hence the formula is of limited practical significance. However, it has two important implications:

1) The <u>mutual inductance is a purely geometrical factor</u> which depends only on the sizes, shapes and relative orientations of the loops.
2) Interchanging the subscripts 1 and 2 yields the same integral, which implies $M_{12} = M_{21} = M$

Now, if I_1 varies, there will be an induced emf in loop 2 given by

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}$$

Unit of inductance: From the above equation, the unit of inductance is volt second per ampere, which is called Henry (H, plural Henries).

More importantly, a change in the current will induce a back emf on loop 1 itself to oppose the change, which is obviously proportional to the rate of change of current as well. The proportionality constant L defined by

$$\mathcal{E}_1 = -L\frac{dI_1}{dt}$$

is called the self-inductance.

Maxwell's Correction

With the Faraday's law, the set of equations now reads

$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \end{cases}$$

If you study them carefully, you will realize that something is wrong!!
Look at the fourth equation, and take divergence of both sides:

$$\nabla \cdot \mathbf{J} = \frac{1}{\mu_0} \nabla \cdot \left(\nabla \times \mathbf{B} \right) = 0$$

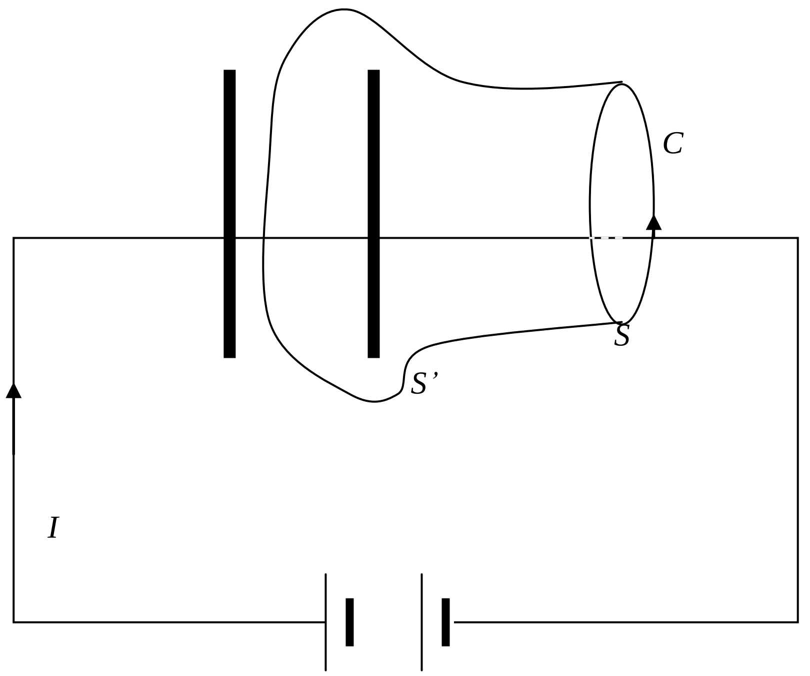
However, from the continuity equation: $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$

which is in general non-zero in electrodynamics.

Similarly, consider the Ampere's law in integral form:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\text{enc}}$$

The current enclosed by *C* is not well defined since different choices of *S* may yield different I_{enc}. This is, of course, also due to the fact that $\nabla \cdot \mathbf{J} \neq 0$ in general.
For instance, consider the following set up of charging up a capacitor:



When the capacitor is being charged up, a current is flowing in the direction shown. Positive and negative charges are being accumulated on the left and right plate of the capacitor, respectively. In between the plates, the electric field is increasing, but there is no current.

Consider the amperian loop *C*, which is assumed to be "flat" for simplicity. If Ampere's law is applied on the loop, and the flat surface *S* is used to calculate I_{enc}, one obtains

$$I_{\text{enc}} = I \,.$$

However, if the curved surface *S'* is chosen, which does not intersect with the wire, then

$$I_{\text{enc}} = 0 \,.$$

Hence, we know that something is missing on the right hand side of the Ampere's law, which, together with $\mu_0 \mathbf{J}$, gives a zero divergence.

Notice that from the continuity equation and Gauss' law:

$$\begin{aligned} \nabla \cdot \mathbf{J} &= -\frac{\partial \rho}{\partial t} \\ &= -\frac{\partial}{\partial t}\left(\varepsilon_0 \nabla \cdot \mathbf{E}\right) \\ &= -\nabla \cdot \left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \\ \Rightarrow \nabla \cdot \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) &= 0 \end{aligned}$$

The second term is called the displacement current:

$$\mathbf{J}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

though it is misleading since it has nothing to do with the flow of charges.

Maxwell proposed that the missing term in the Ampere's law is

$\mu_0 \mathbf{J}_d = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \,.$$

In integral form, it reads

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathbf{enc}} + \mu_0 \varepsilon_0 \int \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{a}$$

By adding this "maxwell's correction term", the conservation of charges is restored. The ambiguity in the definition of current enclosed is also solved by including the displacement current. It turns out that it is the sum of real current and displacement current that is unchanged no matter what surface one chooses.

Also note the parallel between the modified Ampere's law and the Faraday's law, in which a changing magnetic field induces an electric field. Here,

A changing electric field induces a magnetic field

Hence there are two sources of magnetic field, viz., $\mathbf{J}$ and $\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$. The second contribution $\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ is difficult to observe as $\mu_0 \varepsilon_0 \approx 10^{-17}$, which is very small, unless the electric field is changing very rapidly. Maxwell derived this term relying solely on mathematics. It was later verified experimentally by the observation of electromagnetic waves it predicted.

Maxwell's Equations

The set of four equations now becomes

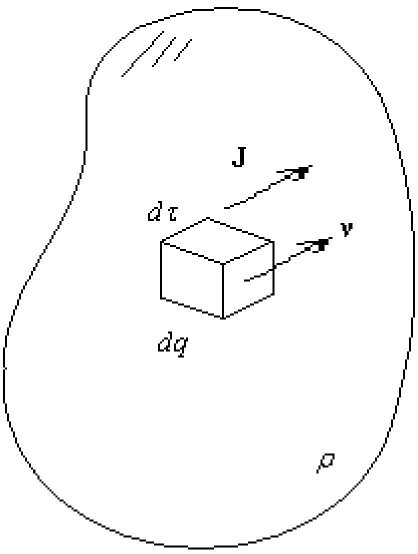
$$\begin{cases} \nabla \cdot \mathbf{E} = \rho / \varepsilon_0 & \text{Gauss' law} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} & \text{Faraday's law} \\ \nabla \cdot \mathbf{B} = 0 & \text{No name} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} & \text{Ampere's law with Maxwell's correction} \end{cases}$$

which are called the **Maxwell's equations**.

Energy of the Electromagnetic Field and the Poynting's Theorem

First of all, what is energy?
Here is Richard Feynman's view:
In physics today, we have no knowledge of what energy ***is***.
However, there are formulas for calculating some numerical quantity, and when we add it all together it always gives the same number.
It is an abstract thing in that it does not tell us the mechanism or the reasons for the various formulas.
In the gradual increase in the complexity of the world we study, we find more and more terms to be added so that the sum is conserved.

Consider a general charge distribution ρ, in which the charges may be moving. Hence ρ depends on both position and time. The motion of the charges give rise to a current density **J** as well, where $\mathbf{J} = \rho\mathbf{v}$.



In the absence of EM field, the total mechanical energy (KE plus PE due to other forces) of all the particles inside $\mathcal{V}$, $W_{\text{mech}}(\mathcal{V})$, is conserved:

$$\frac{dW_{\text{mech}}(\mathcal{V})}{dt} = 0.$$

However, this is obviously wrong when there are electromagnetic forces, which do work on the particles and changes their mechanical energy.

Notice that the magnetic field always does no work. Hence the work done on a small volume element $d\tau'$, carrying charges $dq = \rho d\tau'$, is solely due to the electric field.

The rate of change of $W_{\text{mech}}(\mathcal{V})$, under electromagnetic interaction, equals rate of work done by the E field:

$$\frac{dW_{\text{mech}}(\mathcal{V})}{dt} = \int_{\mathcal{V}} dq\mathbf{E}\cdot\mathbf{v} = \int_{\mathcal{V}} \rho\mathbf{v}\cdot\mathbf{E}d\tau' = \int_{\mathcal{V}} \mathbf{J}\cdot\mathbf{E}d\tau'$$

With the modified Ampere's law,

$$\begin{aligned}\frac{dW_{\text{mech}}(\mathcal{V})}{dt} &= \int_{\mathcal{V}}\left[\frac{1}{\mu_0}(\nabla\times\mathbf{B}) - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}\right]\cdot\mathbf{E}d\tau' \\ &= \int_{\mathcal{V}}\frac{1}{\mu_0}(\nabla\times\mathbf{B})\cdot\mathbf{E}d\tau' - \varepsilon_0\int_{\mathcal{V}}\frac{\partial\mathbf{E}}{\partial t}\cdot\mathbf{E}d\tau' \\ &= \int_{\mathcal{V}}\frac{1}{\mu_0}(\nabla\times\mathbf{B})\cdot\mathbf{E}d\tau' - \frac{\varepsilon_0}{2}\int_{\mathcal{V}}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E})d\tau'\end{aligned}$$

Hence,

$$\frac{dW_{\text{mech}}(\mathcal{V})}{dt} + \frac{\varepsilon_0}{2}\int_{\mathcal{V}}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E})d\tau' = \int_{\mathcal{V}}\frac{1}{\mu_0}(\nabla\times\mathbf{B})\cdot\mathbf{E}d\tau'$$

Consider the right hand side. Recall that

$$\nabla\cdot(\mathbf{E}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{B})$$

Hence

$$\begin{aligned}\int_{\mathcal{V}}\frac{1}{\mu_0}(\nabla\times\mathbf{B})\cdot\mathbf{E}d\tau' &= \int_{\mathcal{V}}\frac{1}{\mu_0}\left[\mathbf{B}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{B})\right]d\tau' \\ &= \frac{1}{\mu_0}\int_{\mathcal{V}}\mathbf{B}\cdot\left(-\frac{\partial\mathbf{B}}{\partial t}\right)d\tau' - \frac{1}{\mu_0}\int_{\mathcal{V}}\nabla\cdot(\mathbf{E}\times\mathbf{B})d\tau' \\ &= -\frac{1}{2\mu_0}\int_{\mathcal{V}}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B})d\tau' - \frac{1}{\mu_0}\int_{S}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}'\end{aligned}$$

where we have used the Faraday's law in the first step and the divergence theorem in the second step to transform the volume integral to a surface integral over S --- the boundary of $\mathcal{V}$.

Rearranging the terms, we then have

$$\frac{dW_{\text{mech}}(\mathcal{V})}{dt} + \frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2}\varepsilon_0 E^2 d\tau' + \frac{d}{dt}\int_{\mathcal{V}}\frac{1}{2\mu_0}B^2 d\tau' = -\int_{S}\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}'\ .$$

This is called the Poynting's theorem, which is the "work-energy theorem" of electrodynamics.

Physical interpretation:
Define

$$W_{\text{em}}(\mathcal{V}) = \int_{\mathcal{V}} \frac{1}{2}\varepsilon_0 E^2 d\tau' + \int_{\mathcal{V}} \frac{1}{2\mu_0} B^2 d\tau' .$$

Then

$$\frac{d}{dt}\left(W_{\text{mech}}(\mathcal{V}) + W_{\text{em}}(\mathcal{V})\right) = -\int_S \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}' ,$$

We have obtained a quantity associated with region $\mathcal{V}$, whose rate of change equals negative the flux of a vector field through its surface. Such a flux-conservative form obviously represents something flowing in space.

Besides, if one integrates over all space, then $\int_S \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}' = 0$ and hence

$$\frac{d}{dt}\left(W_{\text{mech}} + \int_{\text{all space}} \frac{1}{2}\varepsilon_0 E^2 d\tau' + \int_{\text{all space}} \frac{1}{2\mu_0} B^2 d\tau'\right) = 0 .$$

We hence interpret

$$W_{\text{em}} = \int_{\mathcal{V}} \frac{1}{2}\varepsilon_0 E^2 d\tau' + \int_{\mathcal{V}} \frac{1}{2\mu_0} B^2 d\tau'$$

as the energy carried by the electromagnetic fields inside $\mathcal{V}$.

Define $$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}),$$

which is called the **Poynting vector**. It has the physical meaning of **energy flux density**. In other words, $\mathbf{S}\cdot d\mathbf{a}'$ is the energy per unit time crossing the infinitesimal area $d\mathbf{a}'$.

Let u_{mech} and $u_{\text{em}} = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2$ denote the mechanical energy density and field energy density, respectively. In other words,

$$\begin{cases} W_{\text{mech}} = \int_{\mathcal{V}} u_{\text{mech}} d\tau' \\ W_{\text{em}} = \int_{\mathcal{V}} u_{\text{em}} d\tau' \end{cases}$$

Then

$$\frac{d}{dt}\left(\int_{\mathcal{V}} u_{\text{mech}} d\tau' + \int_{\mathcal{V}} u_{\text{em}} d\tau'\right) = -\oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}'$$

$$\Rightarrow \int_{\mathcal{V}} \frac{\partial}{\partial t}\left(u_{\text{mech}} + u_{\text{em}}\right) d\tau' = -\oint_{\mathcal{S}} \mathbf{S}\cdot d\mathbf{a}'$$

By divergence theorem,

$$\frac{\partial}{\partial t}\left(u_{\text{mech}} + u_{\text{em}}\right) + \nabla\cdot\mathbf{S} = 0$$

which is the differential form of Poynting's theorem.

Comparing the above equation with the continuity equation of charges:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{J} = 0$$

one can see that it is just the continuity equation of energy.

Electromagnetic Waves in Vacuum

The Maxwell's equations predict the existence of electromagnetic waves. In vacuum, the Maxwell's equations read

$$\begin{cases} \nabla\cdot\mathbf{E} = 0 \\ \nabla\times\mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla\cdot\mathbf{B} = 0 \\ \nabla\times\mathbf{B} = \mu_0\varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

Taking the curl on both sides of the Faraday's law, we have

$$\nabla\times\left(\nabla\times\mathbf{E}\right) = -\frac{\partial}{\partial t}\nabla\times\mathbf{B}$$

By Ampere's law, $$\nabla\left(\nabla\cdot\mathbf{E}\right) - \nabla^2\mathbf{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

By Gauss' law $$\nabla^2\mathbf{E} = \mu_0\varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

which is the wave equation in 3D.

Similarly, by taking the curl on both sides of the Ampere's law, we have

$$\nabla\times(\nabla\times\mathbf{B}) = \mu_0\varepsilon_0\frac{\partial}{\partial t}\nabla\times\mathbf{E}$$

By Faraday's law, $$\nabla(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = -\mu_0\varepsilon_0\frac{\partial}{\partial t}\frac{\partial\mathbf{B}}{\partial t}$$

Since $\nabla\cdot\mathbf{B}=0$, we have

$$\nabla^2\mathbf{B} = \mu_0\varepsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Therefore, both the E field and B field satisfy the wave equation and admit solution of propagating waves.

Comparing with the wave equation in 1-D $\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$, the speed of EM wave is given by

$$c = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \approx \frac{1}{\sqrt{4\pi\times10^{-7}\times8.85\times10^{-12}}} \approx \frac{1}{\sqrt{1.11\times10^{-17}}} \approx 3.00\times10^8\,\mathrm{ms}^{-1}$$

Maxwell's Equations Inside Matter

Inside matter, there are in general polarization **P** and magnetization **M**. The Gauss' law and the Ampere's law can be re-formulated. For the Gauss' law, the total charge is the sum of free charges and bound charges:

$$\nabla\cdot\mathbf{E} = \frac{1}{\varepsilon_0}(\rho_f + \rho_b)$$

where

$$\rho_b = -\nabla\cdot\mathbf{P}$$

Hence,

$$\nabla\cdot\mathbf{D} = \rho_f$$

where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}\,.$$

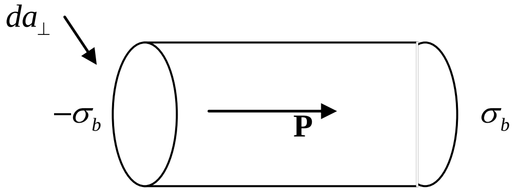
In magnetostatics, we have also learned that on the right hand side of the Ampere's law, the total current consists of two contributions, viz., free currents and bound currents due to magnetization. Hence, you may think that the Ampere's law in electrodynamics should be

$$\nabla\times\mathbf{B} = \mu_0(\mathbf{J}_f + \mathbf{J}_b) + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

where $$\mathbf{J}_b = \nabla \times \mathbf{M}.$$

However, in electrodynamics, there is another contribution to the total current in the above equation.
In electrodynamics, **P** varies with time in general. This means that the charges inside the electric dipoles are moving, giving rise to a current which is called the **polarization current** $\mathbf{J}_p$.

Consider a small piece of matter with polarization **P**, as shown below:



We know that there will the surface bound charges at both ends with density $\sigma_b = P$.

When **P** varies, the net effect is that a current *dI* flows in the direction of **P**. The magnitude of the current is

$$dI = \frac{\partial}{\partial t}\left(\sigma_b da_\perp\right)$$

Hence, the volume current density is

$$\mathbf{J}_p = \frac{dI}{da_\perp}\hat{\mathbf{P}} = \frac{\partial \sigma_b}{\partial t}\hat{\mathbf{P}} = \frac{\partial P}{\partial t}\hat{\mathbf{P}} = \frac{\partial \mathbf{P}}{\partial t}$$

Taking into account the polarization current, the Ampere's law inside matter should be

$$\nabla \times \mathbf{B} = \mu_0\left(\mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p\right) + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J}_f + \mu_0\nabla \times \mathbf{M} + \mu_0\frac{\partial \mathbf{P}}{\partial t} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \left(\frac{1}{\mu_0}\mathbf{B} - \mathbf{M}\right) = \mathbf{J}_f + \frac{\partial\left(\varepsilon_0\mathbf{E} + \mathbf{P}\right)}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

where $$\mathbf{H} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}.$$

The two remaining equations

$$\begin{cases} \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \end{cases}$$

involve no source and are hence unchanged inside matter.
In conclusion, inside matter:

$$\begin{cases} \nabla \cdot \mathbf{D} = \rho_f \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = J_f + \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

The equations are close are providing the constitutive relations, which relate polarization to the E field and magnetization to the B field.
For instance, for linear media,

$$\begin{cases} \mathbf{D} = \varepsilon \mathbf{E} \\ \mathbf{H} = \dfrac{1}{\mu} \mathbf{B} \end{cases}$$

Electromagnetic Waves in Matter

Inside matter with no free charges and currents, the Maxwell's equations become

$$\begin{cases} \nabla \cdot \mathbf{D} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{H} = \dfrac{\partial \mathbf{D}}{\partial t} \end{cases}$$

If the medium is linear, then the equations reduce to

$$\begin{cases} \nabla \cdot \mathbf{E} = 0 \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu\varepsilon \dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

Notice that these are just the Maxwell's equations in vacuum under the transcription $\varepsilon_0 \to \varepsilon,\ \mu_0 \to \mu$.

Hence, the E field and B field satisfy the wave equation

$$\begin{cases} \nabla^2 \mathbf{E} = \mu\varepsilon \dfrac{\partial^2 \mathbf{E}}{\partial t^2} \\ \nabla^2 \mathbf{B} = \mu\varepsilon \dfrac{\partial^2 \mathbf{B}}{\partial t^2} \end{cases}$$

and the speed of light becomes

$$v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} \Big/ \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} = \frac{c}{n}$$

In other words, the speed of light in matter is reduced by a factor

$$n = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}} \, ,$$

which is called the refractive index.

For most materials, $\mu \approx \mu_0$, and $\varepsilon > \varepsilon_0$, hence $n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} = \sqrt{K} > 1$, where K is the dielectric constant. Hence $v < c$.