第5章 基本解与解的基本表达式

1、δ函数

$$\delta(x) = \delta(-x)$$

$$\int_{-\infty}^{+\infty} \delta'(x) f(x)\,dx = -f'(0)$$

$$F[\delta(x)] = 1$$

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\lambda x}\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\lambda x}\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \cos\lambda x\,d\lambda$$

2、泊松方程

① 三维问题

一般提法 $\begin{cases} \Delta_3 u = -f(x,y,z) \\ u|_S = \varphi(x,y,z) \end{cases}$

1° 基本解问题 $\Delta_3 U = \delta(M-M_0)$

基本解 $U(M) = -\dfrac{1}{4\pi r(M,M_0)}$

2° 格林函数问题 $\begin{cases} \Delta_3 G = -\delta(M-M_0) \\ G|_S = 0 \end{cases}$

球内格林函数 $G(M) = \dfrac{1}{4\pi}\left[\dfrac{1}{r(M,M_0)} - \dfrac{R}{\rho_0}\dfrac{1}{r(M,M_1)}\right]$

其中 $\rho_0 = \overline{OM_0}$，$\overline{OM_1} = \dfrac{R^2}{\rho_0}$

3° 解的基本表达式

$$u(M) = -\iint_S \varphi(M_0)\frac{\partial G}{\partial n}\,dS_0 + \iiint_V f(M_0)\,G\,dM_0$$

② 二维问题

1° 基本解问题 $\Delta_2 U = \delta(M-M_0)$

基本解 $U(M) = -\dfrac{1}{2\pi}\ln\dfrac{1}{r}$

2° 圆内格林函数 $G(M) = \dfrac{1}{2\pi}\left[\ln\dfrac{1}{r(M,M_0)} - \ln\left(\dfrac{R}{\rho_0}\dfrac{1}{r(M,M_1)}\right)\right]$

其中 $\rho_0 = \overline{OM_0}$，$\overline{OM_1} = \dfrac{R^2}{\rho_0}$

3° 解的基本表达式

$$u(M) = -\int_l \varphi(M_0)\frac{\partial G(M;M_0)}{\partial n}\,dl + \iint_D f(M_0)\,G(M;M_0)\,dA$$

3、热传导方程

一般提法 $\begin{cases} u_t = a^2\Delta u + f(t,M) \\ u|_{t=0} = \varphi(M) \end{cases}$

1° 基本解问题 $\begin{cases} U_t = a^2\Delta U \\ U|_{t=0} = \delta(M) \end{cases}$

基本解（以三维为例） $U(t,M) = \left(\dfrac{1}{2a\sqrt{\pi t}}\right)^3 \exp\left\{-\dfrac{x^2+y^2+z^2}{4a^2t}\right\}$

2° 解的基本表达式 $u(t,M) = U*\varphi + \int_0^t U(t-\tau,M) * f(\tau,M)\,d\tau$

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4、波动方程

①三维问题　一般提法
$$\begin{cases} u_{tt} = a^2\Delta_3 u + f(t,M) \\ u(0,M) = \varphi(M), \quad u_t(0,M) = \psi(M) \end{cases}$$

1° 基本解问题
$$\begin{cases} U_{tt} = a^2\Delta_3 U \\ U(0,M) = 0, \quad U_t(0,M) = \delta(M) \end{cases}$$

基本解　$U(t,M) = \frac{1}{4\pi a r}\delta(r-at)$

2° 解的基本表达式

$$u(t,M) = \frac{\partial}{\partial t}[U*\varphi] + U*\psi + \int_0^t U(t-\tau,M)*f(\tau,M)d\tau$$

$$u(t,M) = tM_{at}(\psi) + \frac{\partial}{\partial t}[tM_{at}(\varphi)] \qquad (f=0)$$

②一维问题　一般提法
$$\begin{cases} u_{tt} = a^2 u_{xx} + f(t,x) \\ u(0,x) = \varphi(x), \quad u_t(0,x) = \psi(x) \end{cases}$$

1° 基本解问题
$$\begin{cases} U_{tt} = a^2 U_{xx} \\ U(0,x) = 0, \quad U_t(0,x) = \delta(x) \end{cases}$$

基本解　$U(t,x) = \frac{1}{2a}h(at-|x|)$

③二维问题　基本解
$$U(t,M) = \begin{cases} \frac{1}{2\pi a}\frac{1}{\sqrt{a^2t^2-r^2}} & (r \le at) \\ 0 & (r > at) \end{cases}$$