Gaussian Random Vector

Def: Elementary (Standard) Gaussian random vector is a random vector composed of real, mean-zero, unit variance, independent Gaussian r.v.'s.

Note: According to the definition, the pdf and cf of standard Gaussian random vector $\underline{X}$ are

$$f_{\underline{X}}(\underline{x}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}\underline{x}^T\underline{x}}$$

$$\Phi_{\underline{X}}(\underline{w}) = \prod_{i=1}^{n} e^{-\frac{w_i^2}{2}} = e^{-\frac{1}{2}\underline{w}^T\underline{w}}$$

Def: Any random vector that can be generated by a linear/affine transformation of a standard Gaussian random vector is call a Gaussian random vector.

Thm: The cf of a Gaussian random vector $\underline{Y}$ with mean $\underline{m}_Y$ and covariance matrix k_Y is

$$\Phi_{\underline{Y}}(\underline{w}) = e^{i\underline{w}^T\underline{m}_Y - \frac{1}{2}\underline{w}^T k_Y \underline{w}}$$

proof: let $\underline{b} = \underline{m}_Y$, $k_Y = GG^T$, $\underline{X}$ is standard Gaussian.

Then $\underline{Y} = G\underline{X} + \underline{b}$ is Gaussian, with

$$\mathbb{E}\underline{Y} = G\underline{m}_X + \underline{b} = \underline{b} = \underline{m}_Y$$

$$\mathbb{E}\{\underline{Y}_0\underline{Y}_0^*\} = G\,\mathbb{E}\{\underline{X}_0\underline{X}_0^*\}G^T = GG^T = k_Y$$

The cf of $\underline{Y}$ is

$$\Phi_{\underline{Y}}(\underline{w}) = \mathbb{E}\{e^{i\underline{w}^T\underline{Y}}\}$$
$$= \mathbb{E}\, e^{i\underline{w}^T(G\underline{X}+\underline{b})}$$
$$= e^{i\underline{w}^T\underline{b}}\, \mathbb{E}\, e^{i(G^T\underline{w})^T\underline{x}}$$

$$= e^{i\underline{w}^T \underline{b}} \, \Phi_{\underline{X}}(G^T \underline{w})$$

$$= e^{i\underline{w}^T \underline{b}} \, e^{-\frac{1}{2}\underline{w}^T G G^T \underline{w}}$$

$$= e^{i\underline{w}^T \underline{m}_Y - \frac{1}{2}\underline{w}^T k_Y \underline{w}} \qquad \square.$$

Thm: Linear/Affine transformations of Gaussian random vectors are Gaussian random vectors.

proof: Let $\underline{Y}$ be a Gaussian random vector with mean $\underline{m}_Y$ and covariance matrix k_Y, then $\underline{Y}$ can be written as

$$\underline{Y} = G\underline{X} + \underline{b}$$

where $k_X = GG^T$, $\underline{b} = \underline{m}_Y$, $\underline{X}$ is standard Gaussian.

For any linear/affine transformation of $\underline{Y}$,

$$\underline{Z} = H\underline{Y} + \underline{c}$$

we have

$$\underline{Z} = (HG)\underline{X} + (H\underline{b} + \underline{c})$$

$\therefore$ $\underline{Z}$ is Gaussian. $\square$.

Thm: The pdf of a Gaussian random vector $\underline{Y}$ with nonsingular k_Y is

$$f_{\underline{Y}}(\underline{y}) = \frac{1}{(\sqrt{2\pi})^n \sqrt{\det(k_Y)}} e^{-\frac{1}{2}(\underline{y}-\underline{m}_Y)^T k_Y^{-1} (\underline{y}-\underline{m}_Y)}$$

proof: $f_{\underline{Y}}(\underline{y}) = \mathcal{F}^{-1}\{\Phi_{\underline{Y}}(\underline{w})\}$

$$= \frac{1}{(2\pi)^n} \int \cdots \int_{\mathbb{R}^n} e^{-i\underline{w}^T \underline{y}} \, e^{i\underline{w}^T \underline{m}_Y - \frac{1}{2}\underline{w}^T k_Y \underline{w}} \, d\underline{w}$$

$$= \frac{1}{(2\pi)^n} \int \cdots \int_{\mathbb{R}^n} e^{-i\underline{v}^T [A^{-1}(\underline{y}-\underline{m}_Y)]} \, e^{-\frac{1}{2}\underline{v}^T \underline{v}} \frac{d\underline{v}}{|\det(A^T)|}$$

$(\underline{v} = A^T \underline{w}, \; k_Y = AA^T)$

$$= \frac{1}{(\sqrt{2\pi})^n |\det(A)|} \left\{ \int\cdots\int_{\mathbb{R}^n} \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{1}{2}[\underline{v} + iA^{-1}(\underline{y}-\underline{m}_Y)]^2} d\underline{v} \right\} \cdot$$

$$e^{-\frac{1}{2}(\underline{y}-\underline{m}_Y)^T A^{-T} A^{-1} (\underline{y}-\underline{m}_Y)}$$

$$= \frac{1}{(\sqrt{2\pi})^n \sqrt{\det(K_Y)}} e^{-\frac{1}{2}(\underline{y}-\underline{m}_Y)^T K_Y^{-1} (\underline{y}-\underline{m}_Y)}$$

$\square$.

Thm: The pdf of Gaussian random vector $\underline{Y}$ with singular K_Y is

$$f_{\underline{Y}}(\underline{y}) = f_{\underline{Y}_a}(\underline{y}_a) \cdot \delta(\underline{y}_b - [\underline{m}_b + BA^{-1}(\underline{y}_a - \underline{m}_a)])$$

where $\underline{Y} = \begin{pmatrix} \underline{Y}_a \\ \underline{Y}_b \end{pmatrix}$, $\underline{Y}_a \in \mathbb{R}^m$, $\underline{Y}_b \in \mathbb{R}^{n-m}$, and $m = \text{rank}(K_Y)$.

suppose K_{Y_a} is nonsingular (otherwise, use a permutation $\underline{Z} = P\underline{Y}$),

write $K_Y = E\Lambda E^T = H_m H_m^T$, with $H_m = (\sqrt{\lambda_1}\underline{e}_1, \cdots, \sqrt{\lambda_m}\underline{e}_m)_{n\times m}$,

write $H_m = \begin{pmatrix} A \\ B \end{pmatrix}$, $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{(n-m)\times m}$,

and $\underline{m}_Y = \begin{pmatrix} \underline{m}_a \\ \underline{m}_b \end{pmatrix}$, $\underline{Y}_a \sim N(\underline{m}_a, AA^T)$

proof: Let $\underline{X}$ be a m-dimensional standard Gaussian random vector,

then we can write $\underline{Y} = H_m \underline{X} + \underline{m}_Y$, because

$$H_m \underline{X} + \underline{m}_Y \sim N(\underline{m}_Y, K_Y)$$

$$\because \begin{cases} \underline{Y}_a = A\underline{X} + \underline{m}_a, & A \text{ nonsingular} \\ \underline{Y}_b = B\underline{X} + \underline{m}_b \end{cases}$$

$$\therefore \begin{cases} \underline{Y}_a \sim N(m_a, AA^T), \\ \underline{Y}_b = BA^{-1}(\underline{Y}_a - \underline{m}_a) + \underline{m}_b \end{cases}$$

$$\therefore f_{\underline{Y}}(\underline{y}) = f_{\underline{Y}_a}(\underline{y}_a) \cdot \delta(\underline{y}_b - [\underline{m}_b + BA^{-1}(\underline{y}_a - \underline{m}_a)])$$

$\square$.

Thm: $\underline{X}_1, \underline{X}_2$ are Gaussian random vectors, then

$$\underline{X}_1, \underline{X}_2 \text{ are uncorrelated} \iff \underline{X}_1 \perp\!\!\!\perp \underline{X}_2$$

proof: $\underline{X}_1, \underline{X}_2$ uncorrelated

$$\iff K_{\underline{X}_1 \underline{X}_2^*} = 0$$

$$\iff \underline{X} = (\underline{X}_1^T, \underline{X}_2^T)^T \sim N\left(\begin{pmatrix} \underline{m}_{X_1} \\ \underline{m}_{X_2} \end{pmatrix}, \begin{pmatrix} K_{X_1} & 0 \\ 0 & K_{X_2} \end{pmatrix} \right)$$

$$\iff f_{\underline{X}}(\underline{x}) = \frac{1}{(\sqrt{2\pi})^{n_1+n_2} \sqrt{\det K_{X_1} \cdot \det K_{X_2}}} e^{-\frac{1}{2}(\underline{x}_1^T - \underline{m}_{X_1}^T, \underline{x}_2^T - \underline{m}_{X_2}^T) \begin{pmatrix} K_{X_1}^{-1} & \\ & K_{X_2}^{-1} \end{pmatrix} \begin{pmatrix} \underline{x}_1 - \underline{m}_{1} \\ \underline{x}_2 - \underline{m}_{2} \end{pmatrix}}$$

$$= \frac{1}{(\sqrt{2\pi})^{n_1} \sqrt{\det K_{X_1}}} e^{-\frac{1}{2}(\underline{x}_1 - \underline{m}_{X_1})^T K_{X_1}^{-1} (\underline{x}_1 - \underline{m}_{X_1})} \cdot$$

$$\frac{1}{(\sqrt{2\pi})^{n_2} \sqrt{\det K_{X_2}}} e^{-\frac{1}{2}(\underline{x}_2 - \underline{m}_{X_2})^T K_{X_2}^{-1} (\underline{x}_2 - \underline{m}_{X_2})}$$

$$\iff \underline{X}_1 \perp\!\!\!\perp \underline{X}_2$$

$\square$.