Chap. 3 Hyperbolic Conservation Laws

用随体导数可以帮助理解 hyperbolic conservation law:

Hyperbolic conservation law:

每个双曲型的方程,都对应着一个物理量守恒。

令 $\frac{D}{Dt}=0$, 即某系统量守恒,

则

$$\square_t+\vec{v}\cdot\square_x=0$$

这就是一个双曲型方程.

1. Shock formation

linear advection equation:
$$\begin{cases} u_t+au_x=0 \\ u(x,0)=u_0(x) \end{cases}$$

the solution is $u(x,t)=u_0(x-at)$

Inviscid Burger's equation:
$$\begin{cases} u_t+uu_x=0 \\ u(x,0)=u_0(x) \end{cases}$$

characteristic line:
$$\begin{cases} \frac{dx}{dt}=u(x,t)=u_0(\overline{X}) \\ x(0)=\overline{X} \end{cases}$$

$$\frac{du}{dt}=u_t+u_x\cdot\frac{dx}{dt}=u_t+u_xu=0$$

case 1: $u_0(x)\nearrow$, continuous

rarefaction

case 2: $u_0(x)\searrow$ (C^∞)

shock wave

2. Elementary waves for the inviscid Burgers' equation.

Riemann Problem: $\begin{cases} u_t + u u_x = 0 \\ u_0(x) = \begin{cases} u_-, & x<0 \\ u_+, & x>0. \end{cases} \end{cases}$

(centered rarefaction wave)

Case 1: $(u_- < u_+)$

$x \sim ax$, $t \sim at$,

guess $u(x,t) = u_c(\frac{x}{t})$: $\xi = \frac{x}{t}$ (self-similar parameter)

then $\frac{\partial}{\partial t} = \frac{d}{d\xi} \cdot \frac{\partial \xi}{\partial t} = -\frac{\xi}{t}\frac{d}{d\xi}$,

$\frac{\partial}{\partial x} = \frac{d}{d\xi}\frac{\partial \xi}{\partial x} = \frac{1}{t}\frac{d}{d\xi}$.

after substitution, we get

$(u-\xi)u' = 0$ $(t \neq 0)$

the $u = \xi$ or $u = \text{constant}$.

*note: Case 1 gives a continuous piecewise smooth solution based on constant state and centered rarefaction wave, which are the two smooth solutions to ~~[illegible]~~

$u_t + f'(u) u_x = 0$

Case 2: Shock wave

Weak solution:

for equation

~~$u_t + f(u)_x = 0$~~,

$-\xi u_\xi + (f(u))_\xi = 0$,

- if $u(\xi)$ is discontinuous at $\xi = w$, and for any $\varepsilon > 0$, $\int_{w-\varepsilon}^{w+\varepsilon} [-\xi u_\xi + (f(u))_\xi] d\xi = 0$
- Then we call $u(\xi)$ is a weak solution for the equation.

$(u_- > u_+)$ (entropy condition)

travelling wave: let $\eta = x - st$. $(s \in \mathbb{R})$

$\begin{cases} u_t + u u_x = \varepsilon u_{xx} \\ \lim_{\eta \to \pm\infty} u(\eta) = u_\pm \end{cases}$

for solution $u(x,t) = u(\eta)$

$-su' + uu' = \varepsilon u''$

intergation: $-su + \frac{u^2}{2} = \varepsilon u' + C$

when $\eta \to \pm\infty$, $C = -su_- + \frac{u_-^2}{2} = -su_+ + \frac{u_+^2}{2}$

Hence, $-s[\![u]\!] + [\![\frac{u^2}{2}]\!] = 0$, and $s = \frac{u_+ + u_-}{2}$

jumping (跌度)

Rankine-Hugoniot admissibility condition:

it represents a relation between the position w of a discontinuity and the corresponding jump quantity.

$u' < 0$

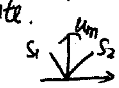
plane analysis

Except case 1, 2, there's no other solutions based on R and S:

1° R & R:

$\xi_1 \neq \xi_2$, the function cannot be continuously connected by constant state.

2° S & S:



$\because u_m < s_1$

$u_m > s_2$, $s_1 < s_2$.

$\therefore$ it's impossible.

3° R & S

$\xi_1 = u_m$

$u_m > s$, $s > \xi_1$

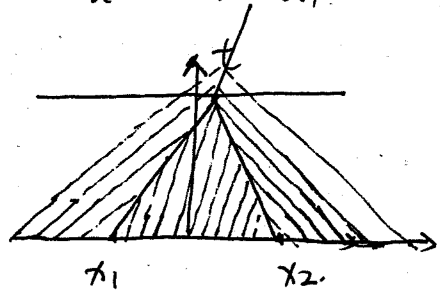
$\therefore$ it's impossible.

*note: Case 2 gives a piecewise smooth solution based on constant state and (self-similar) ~~[illegible]~~ in the sense of weak solution.

Interaction of elementary waves for the inviscid Burgers's equation

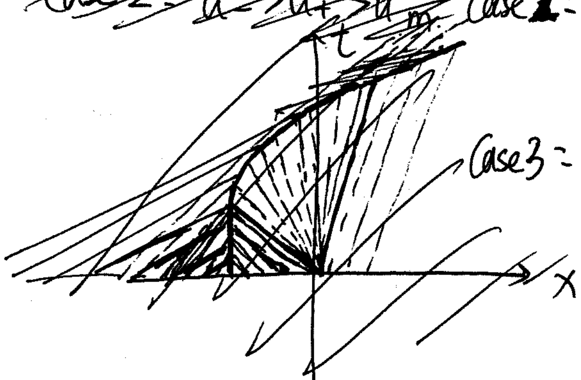
$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \\ u_0(x) = \begin{cases} u_- , & x < x_1 \\ u_m , & x_1 < x < x_2 \\ u_+ , & x > x_2 \end{cases} \end{cases}$$

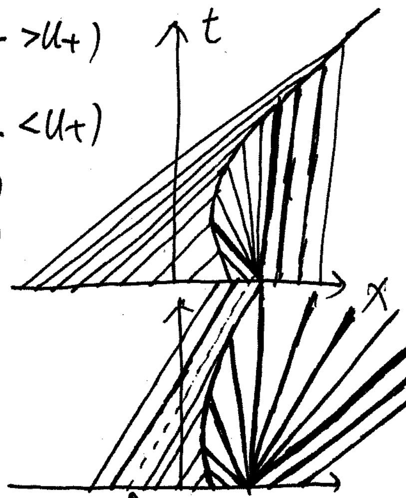
Case 1 : $u_- > u_m > u_+$: $S_1 S_2 \to S_3$



~~Case 2 : $u_- > u_+ > u_m$~~ Case 2 : $S_1 R_1 \to \begin{cases} S_2 & (u_- > u_+) \\ R_2 & (u_- < u_+) \end{cases}$

Case 3 : $RS \to \begin{cases} S & (u_- > u_+) \\ R & (u_- < u_+) \end{cases}$





Case 4 : $RR \to R$

3. elementary waves in a polytropic gas (多方气体) (p-system)

$$\begin{cases} \rho_t + (\rho u)_x = 0 \\ (\rho u)_t + (\rho u^2 + p)_x = 0 \end{cases} \qquad (p = \rho^\gamma,\ 1 \le \gamma < 3)$$

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① for a linear PDE sys. $U_t + AU_x = 0$, it is <u>hyperbolic</u> if A has distinct eigenvalue and complete set of eigenvectors.

If so, from linear algebra, we know A is similar to a diagonal matrix:

$$PAP^{-1} = \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_n).$$

② for a first order quasilinear PDE sys:

$$\sum_{i=1}^{n} a_i(u,x)\frac{\partial u}{\partial x_i} + b(u,x) = 0$$

quasilinear hyperbolic conservation laws :

$U_t + (F(U(x,t)))_x = 0$, and $U(x,t) \in \mathbb{R}^n$, $A(U) = \nabla F(U)$ is a diagonalizable $n \times n$ matrix.

perturbation:

(ρ_0, u_0) is a ground state, which is constant both in space and time.

Denote the perturbation from the ground state as $(\rho, u) = (\rho_0, u_0) + (\hat{\rho}, \hat{u})$.

then we can get the system linearized :

$$\tilde{U}_t + A(U_0)\tilde{U}_x = 0$$

$$A(U) = \begin{bmatrix} 0 & 1 \\ -u^2+p'(\rho) & 2u \end{bmatrix}$$

find the eigen-value :

$$\begin{bmatrix} \lambda_\pm & -1 \\ u^2-p'(\rho) & \lambda_\pm - 2u \end{bmatrix} \begin{bmatrix} 1 \\ \lambda_\pm \end{bmatrix} = 0$$

$$\lambda_\pm = u \pm \underbrace{\sqrt{p'(\rho)}}_{c}$$

(if $\lambda_+ \neq \lambda_-$, then we call the hyperbolic system as strictly hyperbolic.).

then $P = \begin{bmatrix} \frac{1}{\sqrt{1+\lambda_+^2}} & \frac{1}{\sqrt{1+\lambda_-^2}} \\ \frac{\lambda_+}{\sqrt{1+\lambda_+^2}} & \frac{\lambda_-}{\sqrt{1+\lambda_-^2}} \end{bmatrix}$

Lagrangian coordinate : *note : we do the transform to simplify the equation

$$\begin{cases} \tau = t \\ \xi = \end{cases}$$

$\xi = \int_{(x_0,0)}^{(x,t)} \rho(\tilde{x},\tilde{t})d\tilde{x} - \rho(\tilde{x},\tilde{t})u(\tilde{x},\tilde{t})d\tilde{t} + \xi(x_0,0)$ ← denotation of a particle [reference : Shengkai Wang's paper]

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial \tau} + \int_{x_0}^{x} -(\rho u)_x dy \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \tau} + (-\rho u)\frac{\partial}{\partial \xi}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x} + \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} = \rho\frac{\partial}{\partial \xi}$$

$$\begin{cases} V_\tau - u_\xi = 0 \\ u_\tau + p(V)_\xi = 0 \end{cases} \qquad \begin{bmatrix} V \\ u \end{bmatrix}_\tau + \begin{bmatrix} 0 & -1 \\ p'(V) & 0 \end{bmatrix}\begin{bmatrix} V \\ u \end{bmatrix}_\xi = 0$$

$(V = \frac{1}{\rho})$ specific volumne 比容 : $\lambda_\pm = \pm\sqrt{-p'(V)}$

Riemann variables. *note : we do the transform to "diagonalize" the equation.

$$r = u - \int \lambda dV = u + \frac{2\sqrt{\gamma}}{\gamma-1} V^{\frac{1-\gamma}{2}} \qquad (\lambda = \sqrt{-p'(V)} = \sqrt{\gamma V^{-(\gamma+1)}})$$

$$\begin{cases} r = u - \int\lambda dV = u + \frac{2\sqrt{\gamma}}{\gamma-1} V^{\frac{1-\gamma}{2}} \\ s = u + \int\lambda dV = u - \frac{2\sqrt{\gamma}}{\gamma-1} V^{\frac{1-\gamma}{2}} \end{cases} \qquad \begin{cases} u = \frac{r+s}{2} \\ V = \left(\frac{\gamma-1}{4\sqrt{\gamma}}(r-s)\right)^{\frac{\gamma-1}{2}} \end{cases}$$

$$\frac{\partial r}{\partial \tau} = u_\tau - \lambda V_\tau = -p_\xi - \lambda u_\xi = \lambda(\lambda V_\xi - u_\xi)$$

$$\frac{\partial r}{\partial \xi} = u_\xi - \lambda V_\xi$$

$$\begin{cases} \frac{\partial r}{\partial \tau} + \lambda\frac{\partial r}{\partial \xi} = 0 \\ \frac{\partial s}{\partial \tau} - \lambda\frac{\partial s}{\partial \xi} = 0 \end{cases} \qquad \left(s = u + \int\lambda dV = u - \frac{2\sqrt{\gamma}}{\gamma-1} V^{\frac{1-\gamma}{2}} \right)$$

$$\begin{cases} r_\tau + \lambda(r-s) r_\xi = 0 \\ s_\tau - \lambda(r-s) s_\xi = 0 \end{cases}$$

$\Leftarrow \lambda(r-s)$ means λ is a function of $(r-s)$. ($V\uparrow$, $\lambda\Downarrow$)
riemann variable 是对 u, v 的替代, 代表信息的传播。左式是 linear advection equation 的形式, 对确定的 $r-s$, λ 就(波动)是波传播的速度, 即声速

domain of dependence: $D = \{\xi_0 - \lambda_{max}\tau_0, \xi_0 + \lambda_{max}\tau_0\}$ for point (ξ_0, τ_0)

range of influence: $R = \{(\xi, \tau) \mid |\xi - \xi_0| \le \lambda_{max}|\tau - \tau_0|\}$

the velocity of r is λ, and the velocity of s is $(-\lambda)$. since $\lambda = \sqrt{-p'(v)} > 0$, the characteristic line of r always goes forward, and that of s always goes backward.

Elementary waves

$$\begin{cases} v_\tau - u_\xi = 0 \\ u_\tau + p(v)_\xi = \nu u_{\xi\xi} \end{cases}$$

TW solution: $\eta = \xi - s\tau$

$$\begin{cases} -s v' - u' = 0 \\ -s u' + p(v)' = \nu u'' \end{cases} \quad ①$$

$\Rightarrow$ R-H relation

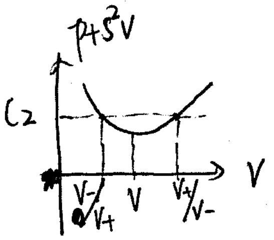
$$\begin{cases} -s[\![v]\!] - [\![u]\!] = 0 \\ -s[\![u]\!] + [\![p]\!] = 0 \end{cases} \Rightarrow -s^2[\![v]\!] = [\![p]\!]$$

$\therefore s_\pm = \pm\sqrt{-\frac{[\![p]\!]}{[\![v]\!]}}$, $[\![u]\!] = -s[\![v]\!]$

S_+: forward shock.

from ①

$$\begin{cases} -sv - u = c_1 \\ -su + p(v) = \nu u' + \tilde{c}_2 \end{cases}$$

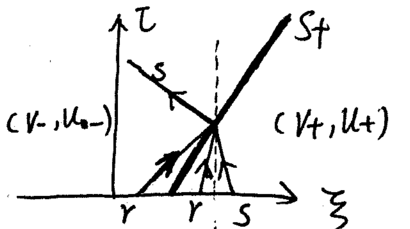


$\therefore s^2 v + p = -\nu s v' + c_2$.

$\therefore -\nu s v' = (s^2 v + p) - (s^2 v_- + p_-) < 0$ (according to convexity of p)

$\therefore v_- < v_+$ (entropy condition)

$\therefore s = \sqrt{-\frac{[\![p]\!]}{[\![v]\!]}}$, $[\![u]\!] = -\sqrt{-[\![p]\!][\![v]\!]}$



(there are 4 characteristic lines in the graph:

$$r: \begin{cases} \frac{d\xi}{d\tau} = \lambda_+(v_-) \\ \frac{d\xi}{d\tau} = \lambda_+(v_+) \end{cases} \quad (\lambda_+(v_-) > s > \lambda_+(v_+)$$

$$s: \begin{cases} \frac{d\xi}{d\tau} = \lambda_-(v_-) \\ \frac{d\xi}{d\tau} = \lambda_-(v_+) \end{cases}$$

they are lines because v on each side of S_+ is constant

S_-: backward shock:

$s = -\sqrt{-\frac{[\![p]\!]}{[\![v]\!]}}$, $[\![u]\!] = -\sqrt{-[\![p]\!][\![v]\!]}$

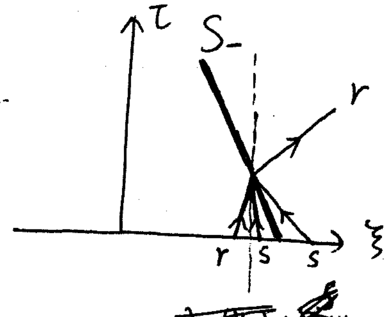
$v_- > v_+$ (entropy condition)

Lax entropy condition: "3-in-1-out."

from the "τ-ξ" graphs of S_+ and S_-,

on a point of S_+/S_-, there are 3 characteristic lines in and 1 out.

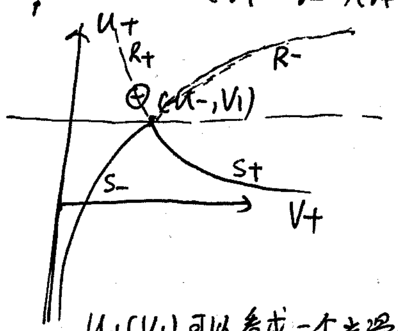
(And "2-in-0-out" for inviscid Burger's equation)

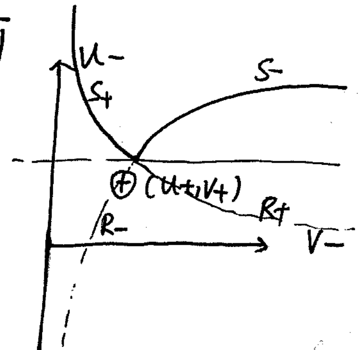


$\lambda_-(v_-) > s > \lambda_-(v_+)$

$u_+ = u_- - \sqrt{-(v_+^{-\gamma} - v_-^{-\gamma})(v_+ - v_-)}$ forward shock.

$u_- = u_+ + \sqrt{-(v_+^{-\gamma} - v_-^{-\gamma})(v_+ - v_-)}$





$u_+(v_+)$ 可以看成一个光滑函数关于 $u=u_-$ 翻折后的图像。

The function is not smooth because absolute value function generated here (say: $\sqrt{f^2(x)}$)

Although $u_+(v_+)$ is an elementary function, the smoothness of it is not guranteed. For elementary functions, we can only say it's continuous. (C^0).

($\sqrt{x^2}$ is not smooth, actually).

R^+: forward rarefaction

take self-similar variable $\eta = \frac{\xi}{\tau}$

we have $\begin{cases} -\eta v' - u' = 0 \\ -\eta u' + p'(v) v' = 0 \end{cases} \Rightarrow \begin{pmatrix} -\eta & -1 \\ p'(v) & -\eta \end{pmatrix} \begin{pmatrix} v' \\ u' \end{pmatrix} = 0$

for non-trivial solution, we have.

$$\det \begin{bmatrix} -\eta & -1 \\ p'(v) & -\eta \end{bmatrix} = \eta^2 + p'(v) = 0$$

$$\therefore \eta = \pm \sqrt{-p'(v)} = \lambda_\pm$$

if $\eta = \lambda_+$ (or $\lambda(v)$), we have a forward rarefaction wave

and $-\lambda v' - u' = 0$

$\because s' = u' + \lambda v'$

$\therefore s' = 0$

explicit expression of the solution:

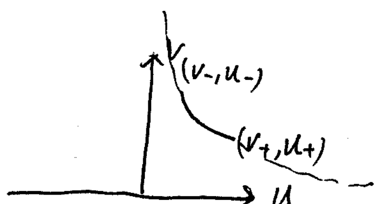
$\because \lambda = \sqrt{\gamma v^{-(\gamma+1)}}$, $\lambda = \eta$

$\therefore v(\eta) = \left(\frac{\eta^2}{\gamma}\right)^{-\frac{1}{\gamma+1}}$ (✓)

$\because ds = du + \lambda dv = 0$

$\therefore du = -\sqrt{\gamma v^{-(\gamma+1)}}\, dv$

$\therefore u(v) = u_0 + \frac{2\sqrt{\gamma}}{\gamma-1}\left(v^{\frac{1-\gamma}{2}} - v_0^{\frac{1-\gamma}{2}}\right) = s_0 + \frac{2\sqrt{\gamma}}{\gamma-1} v^{\frac{1-\gamma}{2}}$ (✓)



Forward rarefaction curve.

R_-: Backward rarefaction

$\eta = \lambda_-$ (or $-\lambda(v)$), we have a backward rarefaction wave.

and $\lambda v' - u' = 0$

$\because r' = u' - \lambda v'$

$\therefore r' = 0$

explicit expression of the solution:

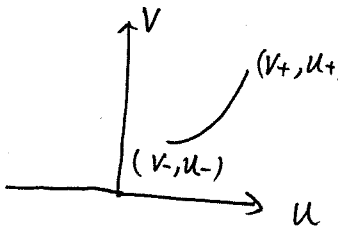
$\because \lambda = -\eta = -\frac{\xi}{\tau}$

$\therefore \nu(\eta) = \left(\frac{\eta^2}{\gamma}\right)^{-\frac{1}{\gamma+1}}$

$\because dr = du - \lambda d\nu = 0$

$\therefore du = \sqrt{\gamma \nu^{-(\gamma+1)}}\, d\nu$

$\therefore u = u_0 - \frac{2\sqrt{\gamma}}{\gamma-1}\left(\nu^{\frac{1-\gamma}{2}} - \nu_0^{\frac{1-\gamma}{2}}\right) = r_0 - \frac{2\sqrt{\gamma}}{\gamma-1}\nu^{\frac{1-\gamma}{2}}$



Appendix:

Proof of existance of Riemann Variable in 2-D systems:

For 1D Eularian equation written in Lagrangian Coordinates:

$$\begin{cases} V_\tau - u_\xi = 0 \\ u_\tau + p(V)_\xi = 0 \end{cases}$$

we want to find variable $r(u,V)$ and $s(u,V)$, such that

$$\exists \mu_1, \mu_2: \begin{cases} r_\tau + \mu_1 r_\xi = 0 \\ s_\tau + \mu_2 s_\xi = 0 \end{cases}$$

that is

$$(r_V, r_u)\left(\begin{pmatrix} V \\ u \end{pmatrix}_\tau + \mu \begin{pmatrix} V \\ u \end{pmatrix}_\xi\right) = 0$$

we already have $\begin{pmatrix} V \\ u \end{pmatrix}_\tau + \begin{pmatrix} 0 & -1 \\ -\lambda^2 & 0 \end{pmatrix}\begin{pmatrix} V \\ u \end{pmatrix}_\xi = 0$, where $\lambda^2 = -p'(V)$

denote the matrix as A.

Then we want to have

$$(r_V, r_u)\left((-A + \mu I)\begin{pmatrix} V \\ u \end{pmatrix}_\xi\right) = 0$$

let $|-A + \mu I| = 0$, we get $\mu = \pm\lambda$

for V_ξ, u_ξ are independent, there must be $(r_V, r_u)(-A + \mu I) = 0$

that is $(r_V, r_u)A = \mu(r_V, r_u)$, (r_V, r_u) is a left-eigenvector of A.

1° if $\mu = \lambda$, then $(r_V, r_u) \parallel (-\lambda, 1)$

that is $\begin{cases} r_V = -\lambda\alpha(u,V) \\ r_u = \alpha(u,V) \end{cases} \Rightarrow r_V + \lambda r_u = 0$

guess a solution with separated variables: $r(u,V) = f(V)g(u)$

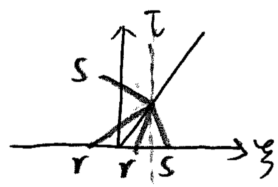
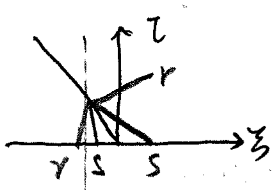
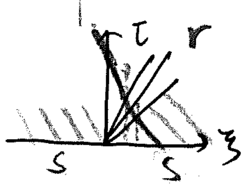
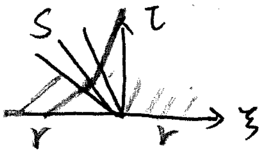
then $\frac{f'(V)}{f(V)\lambda(V)} + \frac{g'(u)}{g(u)} = 0$, let $\frac{f'(V)}{f(V)\lambda(V)} = -1$ (we only want one solution)

then $f(V) = e^{-\int \lambda dV}$, $g(u) = e^u$ (dismiss the constants)

$\therefore r(u,V) = e^{u - \int\lambda dV}$, with $r_\tau + \lambda r_\xi = 0$

2° Similarly, if $\mu = -\lambda$, $s(u,V) = e^{u + \int \lambda dV}$, with $s_\tau - \lambda s_\xi = 0$ $\square$.

Summary

	ξ-τ graph	sound speed & [u] ~~R-H relation~~	entropy condition	R-H relation
1° S_+		$\begin{cases} s=\sqrt{-\frac{[p]}{[v]}} \\ [u]=-\sqrt{-[p][v]} \end{cases}$	$\begin{cases} v_-<v_+ \\ [u]<0 \end{cases}$	$\begin{cases} -s[v]-[u]=0 \\ -s[u]+[p]=0 \end{cases}$
2° S_-		$\begin{cases} s=-\sqrt{-\frac{[p]}{[v]}} \\ [u]=-\sqrt{-[p][v]} \end{cases}$	$\begin{cases} v_->v_+ \\ [u]<0 \end{cases}$	(same with above)
3° R_+		$\lambda(v)=\eta$ $u+\int\lambda dv=\text{const.}$ (on the whole plane)	$v_->v_+$	—
4° R_-		$\lambda(v)=-\eta$ $u-\int\lambda dv=\text{Const.}$ (on the whole plane)	$v_-<v_+$	—

4. Riemann problems.

Review $\begin{cases} v_\tau - u_\xi = 0 \\ u_\tau + p(v)_\xi = \square \end{cases}$ let $\eta = \xi - s\tau$ (TW variable)

the R-H condition is : $\begin{cases} s^2[v] + [p] = 0 & \Rightarrow s = \pm\sqrt{-\frac{[p]}{[v]}} \\ [u] = -s[v] = -\sqrt{[p][v]} \end{cases}$

let $\eta = \frac{\xi}{\tau}$ (self-similar variable)

then $\begin{cases} -\eta v' - u' = 0 \\ -\eta u' + p' = 0 \end{cases} \Rightarrow \eta = \pm\lambda(v)$

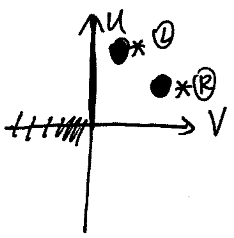
R^+ : ② $\eta = \lambda(v) = \sqrt{\gamma}\cdot v^{-\frac{1+\gamma}{2}}$

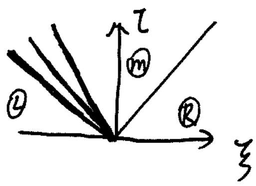
$u = c - \sqrt{\gamma}\,\frac{2}{1-\gamma}\, v^{\frac{1-\gamma}{2}}$

① $s = u + \int \lambda\, dv = c$

R^-

Riemann prob.

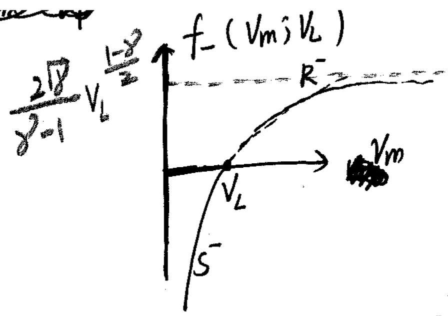


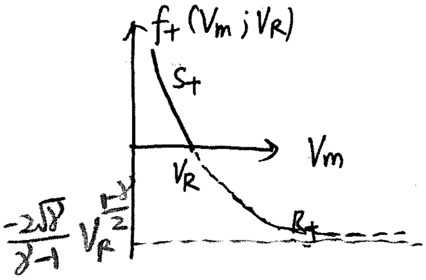


$Ⓛ \xrightarrow{R^-/S^-} ⓜ \xrightarrow{R^+/S^+} Ⓡ$

$f_-(V_m; V_L) = u_m(V_m) = u_L + \begin{cases} -\sqrt{-(p(V_m) - p(V_L))(V_m - V_L)} & V_L > V_m \\ \int_{V_L}^{V_m} \lambda(v)\, dv & V_L < V_m \end{cases}$

$f_+(V_m; V_R) = u_m(V_m) = u_R + \begin{cases} \sqrt{-(p(V_R) - p(V_m))(V_R - V_m)} & V_m < V_R \\ -\int_{V_R}^{V_m} \lambda(v)\, dv & V_m > V_R \end{cases}$

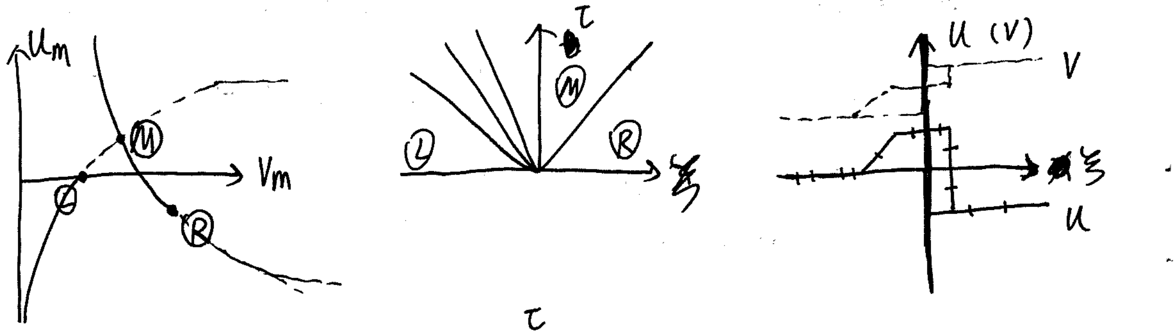




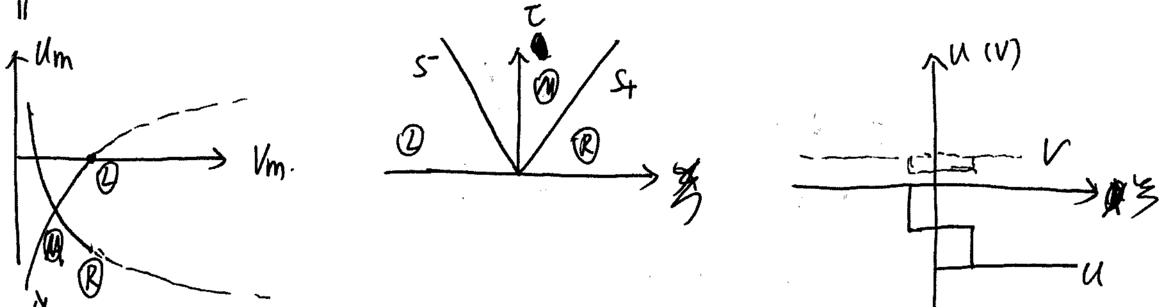
Because $\begin{cases} U_m = U_L + f_-(V_m;V_L) \\ U_m = U_R + f_+(V_m;V_R) \end{cases}$

Hence, (U_m, V_m) should be the intercection of $(f_- + U_L)$ and $(f_+ + U_R)$

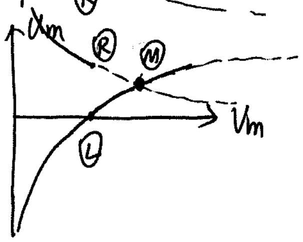
Case 1: $R^- \oplus S^+$



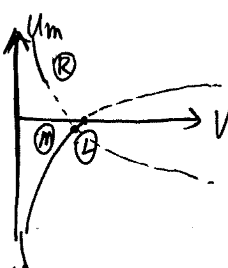
Case 2: $S^- \oplus S^+$



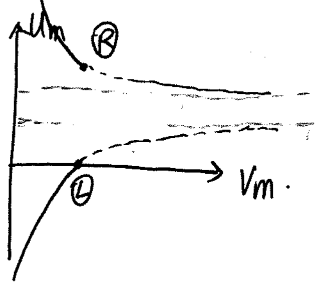
Case 3: $R^- \oplus R^+$



Case 4: $S^- \oplus R^+$



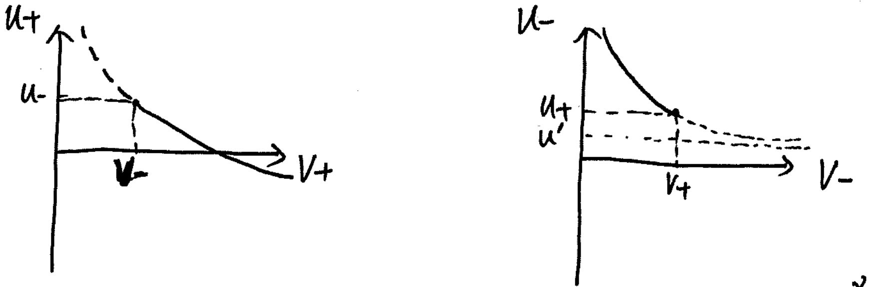
Case 5:



Besides these five cases and special conditions of elementary waves, there is no other solution to the Riemann problem of polytropic gas.

Appendix :

- Proof of the smoothness of $u_+(V_+) / u_-(V_-)$ at point $(u_-, V_-) / (u_+, V_+)$



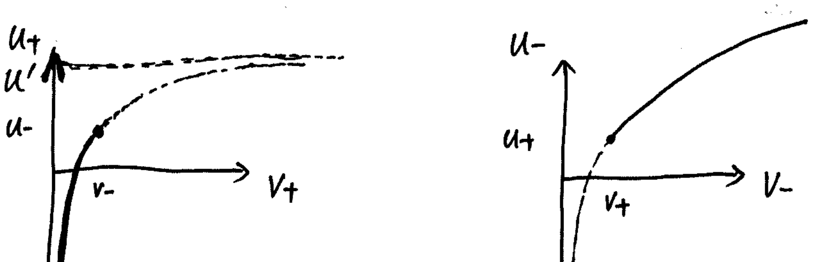
$$u' = u_+ - \frac{2\sqrt{\gamma}}{\gamma - 1} V_+^{\frac{1-\gamma}{2}}$$

--- R_+ — S_+ --- R_+ — S_+

Graph 1. Forward wave curve

$$S_+ : \begin{cases} u_+(V_+) = u_- - \sqrt{-(V_+^{-\gamma} - V_-^{-\gamma})(V_+ - V_-)} \\ u_-(V_-) = u_+ + \sqrt{-(V_-^{-\gamma} - V_+^{-\gamma})(V_- - V_+)} \end{cases} \quad (V_+ > V_-)$$

$$R_+ : u(V) = u_0 + \frac{2\sqrt{\gamma}}{\gamma - 1} \left(V^{\frac{1-\gamma}{2}} - V_0^{\frac{1-\gamma}{2}} \right) \quad (V_+ < V_-)$$



$$u' = u_- + \frac{2\sqrt{\gamma}}{\gamma - 1} V_-^{\frac{1-\gamma}{2}}$$

--- R_- — S_- --- R_- — S_-

Graph 2. Backward wave curve

$$S_- : \begin{cases} u_+(V_+) = u_- - \sqrt{-(V_+^{-\gamma} - V_-^{-\gamma})(V_+ - V_-)} \\ u_-(V_-) = u_+ + \sqrt{-(V_+^{-\gamma} - V_-^{-\gamma})(V_+ - V_-)} \end{cases} \quad (V_+ < V_-)$$

$$R_- : u(V) = u_0 - \frac{2\sqrt{\gamma}}{\gamma - 1} \left(V^{\frac{1-\gamma}{2}} - V_0^{\frac{1-\gamma}{2}} \right) \quad (V_+ > V_-)$$

proof: We first look at $u_+(V_+)$ of forward waves.

1° Continuous (C^0)

the two pieces of $u_+(V_+)$ obviously pass through (V_-, u_-)

2° Continuously differentiable (C^1)

$$S_+ : u_+'(V_+) = - \frac{-(-\gamma V_+^{-\gamma-1})(V_+ - V_-) - (V_+^{-\gamma} - V_-^{-\gamma})}{2\sqrt{-(V_+^{-\gamma} - V_-^{-\gamma})(V_+ - V_-)}}$$

$\therefore U_+'(V_+) = -\frac{1}{2}\gamma V_+^{-\gamma-1}\sqrt{\frac{V_+-V_-}{-(V_+^{-\gamma}-V_-^{-\gamma})}} - \frac{1}{2}\sqrt{\frac{-(V_+^{-\gamma}-V_-^{-\gamma})}{V_+-V_-}}$

$\because \lim\limits_{V_+\to V_-+0} \frac{V_+-V_-}{-(V_+^{-\gamma}-V_-^{-\gamma})} = \lim\limits_{V_+\to V_-+0} \frac{1}{-(-\gamma)V_+^{-\gamma-1}} = \gamma^{-1}V_-^{\gamma+1}$

$\therefore \lim\limits_{V_+\to V_-+0} U_+'(V_+) = -\frac{1}{2}\gamma V_+^{-\gamma-1}\gamma^{-\frac{1}{2}}V_-^{\frac{\gamma+1}{2}} - \frac{1}{2}\gamma^{\frac{1}{2}}V_-^{\frac{-\gamma-1}{2}}$

$= -\gamma^{\frac{1}{2}}V_-^{\frac{-\gamma-1}{2}}$

R_+: $U_+'(V_+) = \frac{2\sqrt{\gamma}}{\gamma-1}\left(\frac{1-\gamma}{2}V_+^{\frac{-1-\gamma}{2}}\right)$

$= -\sqrt{\gamma}\,V_+^{\frac{-\gamma-1}{2}}$

$\therefore \lim\limits_{V_+\to V_-+0} U_+'(V_+) = \lim\limits_{V_+\to V_--0} U_+'(V_+)$

$\therefore$ $U_+(V_+)$ is continuously differentiable at (U_-, V_-)

3° (C^2)

this is true but we don't prove it here. (see Assignment)

$\because$ for S: $U_+(V_+)$ can be seen as a smooth function reflected about $u = U_-$,

$\therefore \lim\limits_{V_+\to V_-+0} U_+'(V_+) = -\lim\limits_{V_+\to V_--0} U_+'(V_+)$, and it is the same with $U_-(V_-)$.

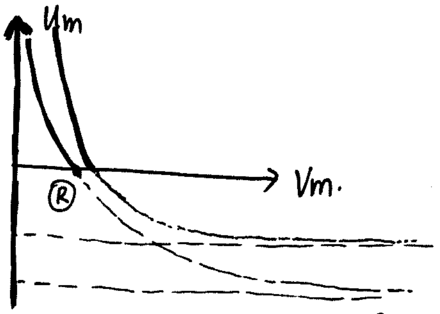
And, for R_+ and R_-, $U(V)$ are symmetric to $u = u_0$

$\therefore U'(V)\big|_{R_+} = -U'(V)\big|_{R_-}$

$\therefore$ the property of $U_+(V_+)$ of forward waves also suits $U_-(V_-)$ of forward waves and $U_+(V_+)$, $U_-(V_-)$ of backward waves. $\square$.

Appendix : 5 Regions of (V_R, U_R) with respect to different solutions.

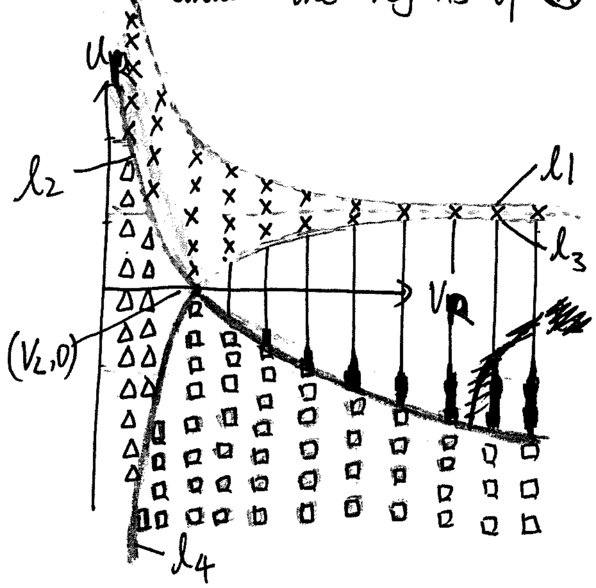
First, we would like to know the figure of forward waves (backward waves) with respect to different (V_R, U_R).



From graph 1, when R moves to the right, the curve is higher; when R moves upward, the curve will keep its shape.

Graph 1. Forward wave curve with different $(V_R, 0)$

Then we can draw the regions of ® with different kinds of solutions.



Graph 2: 5 Regions.

1 : Case 1 $R^- S^+$

□ : Case 2 $S^- S^+$

× : Case 3 $R^- R^+$

Δ : Case 4 $S^- R^+$

: Case 5 Vacuum.

l_1: $U_R(V_R) = \frac{2\sqrt{\gamma}}{\gamma - 1}\left(V_L^{\frac{1-\gamma}{2}} + V_R^{\frac{1-\gamma}{2}}\right)$

l_2: $U_R(V_R) = \frac{2\sqrt{\gamma}}{\gamma - 1}\left(V_R^{\frac{1-\gamma}{2}} - V_L^{\frac{1-\gamma}{2}}\right)$ $(V_R < V_L)$

l_3: $U_R(V_R) = -\frac{2\sqrt{\gamma}}{\gamma - 1}\left(V_R^{\frac{1-\gamma}{2}} - V_L^{\frac{1-\gamma}{2}}\right)$ $(V_R > V_L)$

l_4: $U_R(V_R) = -\sqrt{-(V_R^{-\gamma} - V_L^{-\gamma})(V_R - V_L)}$

denote the ~~upper~~ left half of l_4 as l_{4l}, and the ~~lower~~ right ~~$(V_R < V_L)$~~ half as l_{4r}. Then the boundaries of 5 regions are:

Case 1: $l_3 + l_{4r}$

Case 2: l_4

Case 3: $l_3 + l_2 + l_1$

Case 4: $l_2 + l_{4l}$

Case 5: l_1

meanings of those four lines:

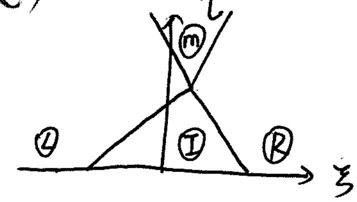
l_1: $U_m(V_m)$ of forward waves and backward waves intersect at infinity

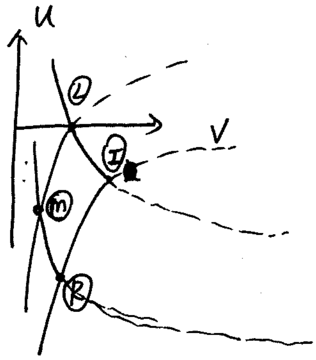
l_{4r}/l_2: $(V_L, 0)$ locates on $U_m(V_m)$ of forward waves.

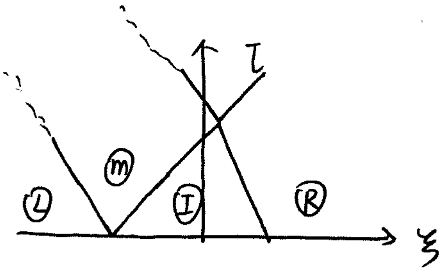
l_{4l}/l_3: ® locates on $U_m(V_m)$ of backward waves.

~~l_{4l} the same with l_3.~~

Interaction of waves in polytropic gas
Additional)







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