Part B:

Hilbert Spaces

Def: Inner product $(\cdot,\cdot): X \times X \to \mathbb{C}$, where X is a linear space, if

1° Additivity: $(x+y, z) = (x,z) + (y,z)$

2° Homogeneity: $(\alpha x, y) = \alpha (x,y)$

3° Symmetry: $(x,y) = \overline{(y,x)}$

4° Positive Definiteness: $(x,x) > 0$ if $x \neq 0$

Def: Inner product space $(X, (\cdot,\cdot))$

Thm: Inner product space is normed with $\|x\| \equiv (x,x)^{\frac{1}{2}}$.

Thm: (Parallelogram Law) X is an normed linear space, then $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$.
$\Leftrightarrow$ X is an inner product space

Def: Hilbert space: Complete inner product space.

Def: Orthogonal $\perp$: $x, y \in$ inner product space X, $(x,y) = 0$.

Thm: (Approximation in Banach and Hilbert spaces)

1° For any point and any (closed) linear subspace in a Hilbert space, there exists a unique point on the subspace that is closest to the point. This point is the orthogonal projection of the original point.

2° For Banach spaces, a closest approximation may not exist.

Cor: For any proper (closed) linear subspace in a Hilbert space, there is a nonzero vector in the Hilbert space orthogonal to the subspace.

Def: <u>orthogonal complement</u>: $M^\perp$ of a subset M of an inner product space X is the set of vectors in X orthogonal to M:

$$M^\perp \equiv \{x \in X : x \perp M\} = \{x \in X : (x,y)=0, \forall y \in M\}.$$

Thm: $M^\perp$ is a closed linear subspace of the inner product space X.

Cor: X is a Hilbert space $\Rightarrow$ $M^\perp$ is a Hilbert space.

Thm: M is a subset of inner product space X:

(1) $\{0\}^\perp = X$, $X^\perp = \{0\}$

(2) M is dense in X $\Longrightarrow$ $M^\perp = \{0\}$

(3) $M \cap M^\perp \subseteq \{0\}$

(4) $M \subset M^{\perp\perp}$

(5) $M^\perp = M^{\perp\perp\perp}$

(6) $M \subset N$ $\Rightarrow$ $N^\perp \subset M^\perp$, $M^{\perp\perp} \subset N^{\perp\perp}$.

Thm: M is a linear subspace of Hilbert space H:

(1) M is dense in H $\Longleftrightarrow$ $M^\perp = \{0\}$

(2) $\overline{M} = M^{\perp\perp}$

Thm: Sum of two orthogonal closed linear subspaces of Hilbert space is also a closed linear subspace.

Thm: (Projection theorem: 1st version)

M is a closed linear subspace of a Hilbert space H $\Rightarrow$ $M + M^\perp = H$.

Def: Orthogonal projection: projection on an inner product space with orthogonal range and null space.

$$\mathcal{R}(P) \perp \mathcal{N}(P)$$

Thm: An orthogonal projection is continuous.

Thm: An orthogonal projection on an inner product space, Its range and null space ~~are closed linear subspaces.~~

(1) are closed linear subspaces;

(2) are orthogonal complement to each other.

$$\mathcal{N}(P) = \mathcal{R}(P)^\perp, \quad \mathcal{R}(P) = \mathcal{N}(P)^\perp.$$

Thm: (Projection theorem: 2^{nd} version)

A closed linear subspace of a Hilbert space is the range of some orthogonal projection; such orthogonal projection is unique.

Def: $1°$ Orthogonal set: $(x_\alpha, x_\beta) = 0, \forall \alpha \neq \beta$

$2°$ Orthonormal set: $(x_\alpha, x_\beta) = \delta_{\alpha\beta}$

Thm: An orthonormal set of points in an inner product space is linearly independent.

Def: Maximal (or, Complete) orthonormal set: The orthonormal set cannot be extended.

Thm: Any orthonormal set can be extended to a maximal orthonormal set.

Def: Orthonormal basis: maximal orthonormal set in Hilbert space.

Thm: (Fourier Series Theorem)

$\{x_n\}$ is an ~~othonormal~~ orthonormal set in a Hilbert space H. TFAE:

(a) $\{x_n\}$ is an orthonormal basis

(b) (Fourier series expansion)

$$x = \sum_n (x, x_n) x_n \quad , \forall x \in H$$

(c) (Parseval equality)

$$(x, y) = \sum_n (x, x_n) \overline{(y, x_n)} \quad , \forall x, y \in H$$

(d)

$$\|x\|^2 = \sum_n |(x, x_n)|^2$$

(e) Any linear subspace of H that contains $\{x_n\}$ is dense in H.

~~Lemma: (The Bessel Inequality)~~

Thm: A Hilbert space has a countable orthonormal basis $\Longleftrightarrow$ it is separable.

Def: <u>Gram-Schmidt orthogonalization process</u>: construction of an orthonormal set from a countable linearly independent set in an inner product space.

Thm: $\{x_n\}$ is a countable linearly independent set in a Hilbert space H,

M is the ~~linear subspace~~ (finitely) spanned by $\{x_n\}$.

if $\dfrac{\left\|\sum_{i=1}^{N} \beta_i x_i\right\|}{\sqrt{\sum_{i=1}^{N} |\beta_i|^2}}$ is bounded and bounded below, $\forall N \in \mathbb{N}$,

then $\forall y \in \overline{M}$ can be uniquely expressed as $y = \sum_n \alpha_n x_n$.

Thm: The set $\{\phi_n(t) = e^{2\pi i n t} : n = 0, \pm 1, \pm 2, \cdots\}$ is a maximal orthonormal set in $L_2[0,1]$.

Def: 1° Two inner product spaces are <u>unitarily equivalent</u> is exist an isomorphism that preserves inner products.

2° The isomorphism mentioned above is called a <u>unitary operator</u>.

Thm: A mapping between two inner-product spaces, is

unitary operator $\Longleftrightarrow$ isometric isomorphism.

Def: <u>unitarily equivalent</u> (operators) :

$$\begin{tikzcd} X_1 \arrow[loop left, "T"] \arrow[r, bend left, "U"] & X_2 \arrow[loop right, "S"] \arrow[l, bend left, "U^{-1}"] \end{tikzcd}$$

T and S are unitarily equivalent if exists unitary operator U s.t. $T = U^{-1} S U$ and $S = U T U^{-1}$.

Def: <u>(Topological) Sum</u> of a collection of mutually orthogonal closed linear subspaces of a Hilbert space is the closure of the linear subspace generated by them.

Thm: (Orthogonal Structure Theorem)

$M = \sum_n M_n$, $M_n \subset H$

1° $\forall x \in M$, $\exists!\, x_n \in M_n$, $\forall n$, s.t. $x = \sum_n x_n$

2° $\forall \{x_n\}$, $\|x_n\|^2 < \infty$, $\exists\, x \in M$, s.t. $x = \sum_n x_n$.

Def: <u>Direct sum</u> of a countable collection of inner product spaces $\{X_i\}$ is

$$X = \oplus_i X_i \equiv \times_i X_i$$

with inner product $(x, y) = \sum_i (x_i, y_i)_i$.

Thm: $\bigoplus_i M_i$ is complete $\Longrightarrow$ $\bigoplus_i M_i$ and $\Sigma_i M_i$ are unitarily equivalent.

Thm: (Riesz Representation Theorem)

On a Hilbert space, any continuous linear functional is equivalent to the inner product with a unique vector in the space.

$$l(\cdot) = (\cdot, y), \quad y \in H$$

Vector y is called the representation of l.

Thm: (Lax-Milgram Theorem)

On a Hilbert space, let $B[u,v]$ be: 1° a sesquilinear functional

2° $\exists a > 0, \forall u, v \in H : \dfrac{|B[u,v]|}{\|u\| \cdot \|v\|} \leq a$

3° $\exists b > 0, \forall u \in H : \dfrac{|B[u,u]|}{\|u\|^2} \geq b$

Any continuous linear functional is equivalent to $B[x, v_0]$ and $\overline{B[u_0, x]}$, for some unique v_0 and u_0.