2. Identities

Thm: Let $X_{\alpha,\beta}$ denote a gamma (α,β) r.v. with pdf $f(x|\alpha,\beta)$, where $\alpha>1$. Then $\forall a,b \in \mathbb{R}$,

$$P(a<X_{\alpha,\beta}<b) = \beta[f(a|\alpha,\beta) - f(b|\alpha,\beta)] + P(a<X_{\alpha-1,\beta}<b)$$

Lemma: (Stein's Lemma) Let $X \sim n(\theta,\sigma^2)$, and $g(\cdot)$ a differentiable function satisfying $\mathbb{E}|g'(X)| < +\infty$. Then,

$$\mathbb{E}[g(X)(X-\theta)] = \sigma^2 \mathbb{E}g'(X)$$

Thm: Let χ_p^2 denote a chi-squared r.v. with p degrees of freedom. For any function $h(x)$,

$$\mathbb{E}h(\chi_p^2) = p\,\mathbb{E}\left(\frac{h(\chi_{p+2}^2)}{\chi_{p+2}^2}\right)$$

, provided the expectations exist.

Thm: (Hwang) Let $g(x)$ be a function with $-\infty < \mathbb{E}g(X) < +\infty$ and $-\infty < g(-1) < +\infty$. Then,

(a) If $X \sim$ Poisson (λ),

$$\mathbb{E}(\lambda g(X)) = \mathbb{E}(X g(X-1))$$

(b) If $X \sim$ negative binomial (r,p),

$$\mathbb{E}((1-p)g(X)) = \mathbb{E}\left(\frac{X}{r+X-1} g(X-1)\right)$$