# Inversion of Fourier Transform by Residues (Chap 22.4.2.2)

Suppose the Fourier transform is defined in some way for all $f \in \mathbb{C}$, in some conditions we can determine $h(t)$ by contour integral methods.

**Thm:** $H(f)$ is analytic in the complex $f$-plane except for a finite number of poles $\nu_d$, $d = 1, \cdots, D$. None of these poles are real. If $\lim_{R \to +\infty} R \cdot \max_{\theta \in [0, 2\pi)} |H(Re^{i\theta})| = 0$, then we can write the invers Fourier transform as

$$h(t) = \begin{cases} 2\pi i \sum_{\{\nu_d : Im\{\nu_d\} > 0\}} Res\left(\nu_d, H(f) e^{i2\pi f t}\right) & (t \geq 0) \\ -2\pi i \sum_{\{\nu_d : Im\{\nu_d\} < 0\}} Res\left(\nu_d, H(f) e^{i2\pi f t}\right) & (t \leq 0) \end{cases}$$

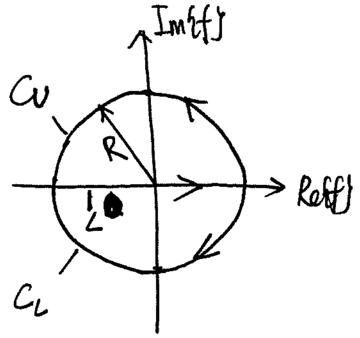
Note: 1° The condition $\lim_{R \to +\infty} R \cdot \max_{\theta \in [0, 2\pi)} |H(Re^{i\theta})| = 0$ is not a necessary cond.

2° Roughly speaking, this thm says that an LTI operator $\mathbb{H}$ is causal, if $H(f)$ has no poles in the lower $f$-plane or on the real line, and goes to zero in all directions in the $f$-plane as $|f| \to \infty$.

Proof:

The assumptions gives

$$H(f) = \frac{G(f)}{\prod_{d=1}^{D} (f - \nu_d)^{m_d}}$$ , where $G(f)$ is analytic (holomorphic)




For contour integral

$$\int_{L \cup C_U} H(f) e^{i2\pi f t} df = \int_{-R}^{R} H(f) e^{i2\pi f t} df + \int_{C_U} H(f) e^{i2\pi f t} df$$

Residue thm gives

$$\lim_{R\to\infty} \int_{LUC_U} H(f) e^{i2\pi ft} df = 2\pi i \sum_{\{v_d : \mathrm{Im}\{v_d\}>0\}} \mathrm{Res}(v_d, H(f)e^{i2\pi ft})$$

where the residues can be calculated by

$$\mathrm{Res}(v_d, H(f)e^{i2\pi ft}) = \frac{1}{(m_d-1)!} \lim_{f\to v_d} \frac{d^{m_d-1}}{df^{m_d-1}} \left[(f-v_d)^{m_d} H(f) e^{i2\pi ft}\right]$$

$\because$ $$\lim_{R\to\infty} \int_{-R}^{R} H(f) e^{i2\pi ft} df = h(t)$$

$$\lim_{R\to\infty} \left| \int_{C_U} H(f) e^{i2\pi ft} df \right| \le \lim_{R\to\infty} \int_0^{\pi} \left| H(Re^{i\theta}) e^{i2\pi Re^{i\theta} t} Re^{i\theta} i \right| d\theta$$

$$\le \lim_{R\to\infty} R\pi \max_{\theta\in[0,\pi)} |H(Re^{i\theta})| = 0 \quad (t \ge 0)$$

$\therefore$ $$h(t) = 2\pi i \sum_{\{v_d : \mathrm{Im}\{v_d\}>0\}} \mathrm{Res}(v_d, H(f)e^{i2\pi ft}) \qquad (t\ge 0)$$

Similarly, for contour integral along $LUC_L$, we get

$$h(t) = -2\pi i \sum_{\{v_d : \mathrm{Im}\{v_d\}<0\}} \mathrm{Res}(v_d, H(f)e^{i2\pi ft}) \qquad (t \le 0) \quad \square.$$