The Kernel Method

For Ordinary Differential Equations with constant coefficients having the following form

$$y'' + by' + cy = g(t), \tag{1}$$

the Kernel Method is often much simpler to apply than Variation of Constants. We state it as a Theorem. (**NOTE:** The coefficient of y'' is 1.)

Theorem 1 (Kernel Method) *Let $K(t)$ be the unique solution of*

$$y'' + by' + cy = 0, \qquad y(0) = 0,\ y'(0) = 1. \tag{2}$$

Then

$$\phi(t) = \int_{t_0}^{t} K(t-s)g(s)ds$$

is the particular solution of (1) satisfying

$$y(t_0) = 0, \qquad y'(t_0) = 0.$$

Proof: Using the Lemma below,

$$\begin{aligned} \phi'(t) &= K(0)g(t) + \int_{t_0}^{t} K'(t-s)g(s)ds \\ \phi''(t) &= K'(0)g(t) + \int_{t_0}^{t} K''(t-s)g(s)ds\ . \end{aligned}$$

Using (2),

$$\phi''(t) + b\phi'(t) + c\phi(t) = g(t) + \int_{t_0}^{t} [K''(t-s) + bK'(t-s) + cK(t-s)]g(s)ds = g(t)\ .$$

Finally,

$$\phi(t_0) = \int_{t_0}^{t_0} K(t-s)g(s)ds = 0 \qquad \text{and} \qquad \phi'(t_0) = \int_{t_0}^{t_0} K'(t-s)g(s)ds = 0$$

Example: Find a particular solution of

$$y'' + y = \sec t \tan t\ . \tag{3}$$

Solution: Any solution of $y'' + y = 0$ has the form

$$y(t) = A\cos t + B\sin t.$$

Imposing the initial conditions in (2), we get $A = 0$ and $B = 1$. Thus $K(t) = \sin t$ and

$$\begin{aligned} \phi(t) &= \int_0^t \sin(t-s)\sec s \tan s\ ds \\ &= \int_0^t [\sin t\cos s - \cos t\sin s]\ \frac{1}{\cos s}\ \frac{\sin s}{\cos s} ds \\ &= \sin t \int_0^t \tan s\ ds - \cos t \int_0^t \tan^2 s\ ds \\ &= \sin t \log(|\sec t|) - \cos t(\tan t - t) \end{aligned}$$

is the particular solution of (3) satisfying $y(0) = y'(0) = 0$.

Lemma 1 *Let*

$$F(t) \doteq \int_a^t g(t,s)ds,$$

where g *and* $\dfrac{\partial g}{\partial t}$ *are continuous on* $(\alpha, \beta) \times (\alpha, \beta)$ *and* $a, t \in (\alpha, \beta)$. *Then*

$$F'(t) = g(t,t) + \int_a^t \frac{\partial g(t,s)}{\partial t} ds.$$