Limit Theorems

(1) Laws of Large numbers

Thm: (Weak Law of Large Numbers, WLLN) 1°

Let $\{X_i\}$ be uncorrelated, $X_i \in \mathcal{L}_2(\mathcal{U}, \mathcal{F}, P)$, with same μ, σ^2. $\overline{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$,

then $\overline{X}_n \xrightarrow{P} \mu$.

proof: $\forall a > 0$, $P\{|\overline{X}_n - \mu| \geq a\} \leq \frac{\sigma^2}{na^2}$ (Chebychev's ineq)

$\because \lim_{n\to\infty} \frac{\sigma^2}{na^2} = 0$

$\therefore \forall a > 0$, $\lim_{n\to\infty} P\{|\overline{X}_n - \mu| \geq a\} = 0$.

$\therefore \overline{X}_n \xrightarrow{P} \mu$ $\square$.

Thm: (Weak Law of Large Numbers, WLLN) 2°

Let $\{X_i\}$ be iid, $X_i \in \mathcal{L}_1(\mathcal{U}, \mathcal{F}, P)$, $\overline{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$,

then $\overline{X}_n \xrightarrow{P} \mu$.

proof:

$$\Phi_{\overline{X}_n}(w) = \left[\Phi_X\left(\frac{w}{n}\right)\right]^n = \left[1 + i\frac{w}{n}\mu + o\left(\frac{1}{n}\right)\right]^n$$

$$\therefore \lim_{n\to\infty} \Phi_{\overline{X}_n}(w) = \lim_{n\to\infty} \left[1 + i\frac{w}{n}\mu + o\left(\frac{1}{n}\right)\right]^{\frac{n}{iw\mu}\cdot iw\mu} = e^{iw\mu} \equiv \Phi_X(w)$$

~~$e^{iw\mu}$ corresponds to pdf~~

Take Fourier transform of $\Phi_X(w)$, we know $e^{iw\mu}$ corresponds to pdf $\delta(x-\mu)$, or cdf $U(x-\mu)$.

$\because \forall w \in \mathbb{R}, \lim_{n\to\infty} \Phi_{\overline{X}_n}(w) = \Phi_X(w) \iff F_{\overline{X}_n}(z) \xrightarrow{w} F_X(z)$
(weak convergence)

$\therefore F_{\overline{X}_n}(z) \xrightarrow{d} U(z-\mu)$, i.e. $\overline{X}_n \xrightarrow{d} \mu$

Since it's a degenerate case, i.e. μ is a constant,

we have $\overline{X}_n \xrightarrow{P} \mu$. $\square$.

Note: 1° Stronger versions of WLLN require $X_i \in \mathcal{L}_2$, which is still applicable in most practical situations.

Thm: (Strong Law of Large Numbers, SLLN) (Kolmogorov)

Let $\{X_i\}$ be i.i.d. r.v.'s in $\mathcal{L}_1(\mathcal{U}, \mathcal{F}, P)$, $\overline{X}_n \equiv \frac{1}{n}\sum_{i=1}^{n} X_i$,

then $\overline{X}_n \xrightarrow{a.s.} \mu$.

Note: 1° LLNs ties together prob. space $(\mathcal{U}, \mathcal{F}, P)$ with relative frequency approach.

2° If $X_i \sim$ Cauchy$(0,1)$, then $\overline{X}_n \longrightarrow$ Cauchy$(0,1)$

(2) Central limit theorems.

Thm: (Central Limit Theorem, CLT) (Lindeberg–Lévy)

Let $\{X_i\}$ be iid r.v.'s in $\mathcal{L}_2(\mathcal{U}, \mathcal{F}, P)$,

then

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$

proof: Denote $Z_n = \sqrt{n}(\overline{X}_n - \mu)$,

then
$$\Phi_{Z_n}(w) = E\{\exp[iw\sum_{i=1}^{n}\frac{X_i-\mu}{\sqrt{n}}]\}$$
$$= [\Phi_{(X_i-\mu)}(\frac{w}{\sqrt{n}})]^n$$
$$= [1 - \frac{w^2}{2n}\sigma^2 + o(\frac{1}{n})]^n$$

$$\therefore \lim_{n\to\infty}\Phi_{Z_n}(w) = \lim_{n\to\infty}[1 - \frac{w^2\sigma^2}{2n} + o(\frac{1}{n})]^{\frac{2n}{w^2\sigma^2}\cdot(\frac{w^2\sigma^2}{2})}$$
$$= e^{-\frac{w^2\sigma^2}{2}} \equiv \Phi_Z(w)$$

(Take Fourier transform of $\Phi_Z(w)$, we get $e^{-\frac{\sigma^2 w^2}{2}}$ corresponds to $N(0, \sigma^2)$.

$\because \forall w \in \mathbb{R}, \lim_{n \to \infty} \Phi_{Z_n}(w) = \Phi_Z(w) \iff F_{Z_n}(z) \xrightarrow{w} F_Z(z)$

$\iff F_{Z_n}(z) \xrightarrow{d} F_Z(z)$

$\therefore \quad Z_n = \sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \square$.)

Thm: (Central Limit Theorem) (Lindeberg - Feller)

Let $\{X_i\}$ be independent r.v.'s in $\mathcal{L}_2(\mathcal{U}, \mathcal{F}, P)$,

then $\sqrt{n}\left(\bar{X}_n - \frac{1}{n}\sum_{i=1}^n \mu_i\right) \xrightarrow{d} N\left(0, \frac{1}{n}\sum_{i=1}^n \sigma_i^2\right)$, if the Lindeberg condition: $\forall \varepsilon > 0, \lim_{n \to \infty} \frac{1}{\sum_{i=1}^n \sigma_i^2} \sum_{i=1}^n \int_{\{z \mid |z - \frac{1}{n}\sum_{i=1}^n \mu_i| > \varepsilon \sqrt{\sum_{i=1}^n \sigma_i^2}\}} \left(z - \frac{1}{n}\sum_{i=1}^n \mu_i\right)^2 dF_{X_n}(z) =$ holds.

Thm: (Central Limit Theorem) (Berry - Essen)

Let $\{X_i\}$ be iid r.v.'s in $\mathcal{L}_3(\mathcal{U}, \mathcal{F}, P)$,

then $$\sup_{z \in \mathbb{R}} \left| F_{Z_n}(z) - F_Z(z) \right| \leq \frac{c}{\sqrt{N}} \cdot \frac{\mathbb{E}|X_1 - \mu|^3}{\sigma^3}$$

where $Z_n \equiv \sqrt{n}\left(\frac{\bar{X}_n - \mu}{\sigma}\right)$, $Z \sim N(0,1)$

c is a constant and $c \in \left[\frac{1}{\sqrt{2\pi}}, 0.8\right)$

Note: 1° CLTs are often used to justify the approx. of a finite sum of r.v.'s by a Gaussian r.v..

2° The Berry - Essen CLT provides the accuracy of approx..

3° With current computing capacity, the importance of approx.'s like CLT is somewhat lessened.

(3) Other limit theorems

Thm: (Continuous mapping thm) If $X_i \xrightarrow{P} X$, $h(\cdot)$ is a continuous fn. on the range of $\{X_i\}$ and X.

then $h(X_i) \xrightarrow{P} h(X)$.

Note: Also holds for d, a.s., mss convergence.

Thm: (Slutsky's theorem)

If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} a$,

then a) $Y_n X_n \xrightarrow{d} aX$

b) $X_n + Y_n \xrightarrow{d} X + a$