Lec. 13 (Linear Space)

$M \subsetneq X$; $\mathcal{D}(U)$; $\mathcal{R}(U)$

proper subspace ; domain ; range

Def: <u>Hamel basis</u> : linearly independent set $B \subseteq X$ that spans X. ($V(B) = X$)

Thm: Every linearly independent subset of a linear space can be extended to a Hamel basis.

Thm: Hamel bases for a linear space have the same cardinal number.

Def: <u>Dimension</u> : $\dim(X)$ = the cardinal number of its Hamel basis.

Thm: $\dim\{\mathcal{N}(L)\} + \dim\{\mathcal{R}(L)\} = \dim(X)$

<u>Matrix representation</u> of finite-dimentional linear transformations:

$$Tx = y \iff [T][x] = [y]$$

Matrix $[T]$ represents T relative to bases $B_1 \subseteq X$ and $B_2 \subseteq Y$, and $[x]$ and $[y]$ are coordinates of x and y under bases B_1 and B_2.

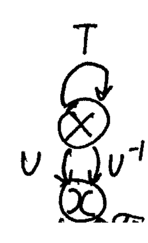
Def: 1° <u>Isomorphically equivalent</u> linear transformations T and $\mathcal{T}$:

$\exists$ isomorphism U, W: $T = W^{-1} \mathcal{T} U$

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, bend right, "U"'] & Y \arrow[d, bend right, "W"'] \\ \mathfrak{X} \arrow[u, bend right, "U^{-1}"'] \arrow[r, "\mathcal{T}"] & \mathcal{Y} \arrow[u, bend right, "W^{-1}"'] \end{tikzcd}$$

2° <u>Similar</u> linear transformations:

$\exists$ isomorphism U: $T = U^{-1} \mathcal{T} U$



Def: (1° <u>Direct sum</u> $X_1 \oplus X_2$: $(x_1, x_2) \in X_1 \oplus X_2$ if $x_1 \in X_1$, $x_2 \in X_2$)

2° <u>Sum</u> $X_1 + X_2$: $x \in X_1 + X_2$ if $\exists x_1 \in X_1, x_2 \in X_2$, s.t. $x_1 + x_2 = x$.

(3° <u>Natural mapping</u> $\phi : X_1 \oplus X_2 \to X_1 + X_2$, by $\phi[(x_1, x_2)] = x_1 + x_2$)

4° <u>Algebraic complement</u> : X_2 is an algbraic complement of X_1 in X, if X_1, X_2 are disjoint and sum up to X.

5° <u>Disjoint</u> linear subspaces : $X_1 \cap X_2 = \{0\}$

Thm: Every linear subspace of a linear space has an algebraic complement.

Def: <u>Projection</u> : linear transformation $P : X \to X$ s.t. $P^2 = P$.

Thm: The range and null space of a projection are algebraic complements of one another.

Thm: Any linear subspace can be the range of some projection.

Note: The projection of any element in the range of a projection is the element itself.

Def: <u>Linear functional</u> : linear $l : X \to F$. (Denote $l(x)$ as $\langle x, l \rangle$)

A linear transformation that maps a linear space into its scalar field

Thm: The algebraic complement of the null space of a linear functional is 1-dimensional

Def: <u>Hyperplane</u> in X determined by l and α : the subspace having a constant value α under linear functional l.

<u>Half-spaces</u>

<u>on one side of the hyperplane</u>

<u>strictly on one side of the hyperplane</u>

Thm: 1° Any linear functional on any linear subspace can be extended to a linear functional on the full space.

2° Any proper linear subspace can be put in the null space of a linear functional.

Def: <u>Algebraic conjugate</u> of a linear space: X^f, the set of all linear functionals on a linear space X.

Note: X^f is a linear space. (over the same scalar field F), where

1° Addition: $l_1 + l_2$: $\langle x, l_1 + l_2 \rangle = \langle x, l_1 \rangle + \langle x, l_2 \rangle$

2° Scalar multiplication: αl : $\langle x, \alpha l \rangle = \alpha \langle x, l \rangle$.

Def: <u>Transpose</u> of a linear transformation: $\langle x, L^T y' \rangle = \langle Lx, y' \rangle$

$$
\begin{tikzcd}
X \arrow[r, bend left, "L"] & Y \\
X^f & Y^f \arrow[l, bend left, "L^T"]
\end{tikzcd}
$$

Thm: 1° L is onto $\Longleftrightarrow$ L^T is 1-1

2° L is 1-1 $\Leftarrow$ L^T is onto.

Note: The lack of symmetry in 2° only occurs in infinite dimensional spaces.