MARKOV CHAINS

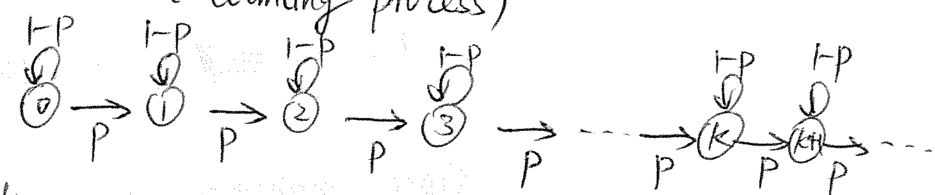
EE464 L25-26-27-28

Discrete-Time Markov Chains

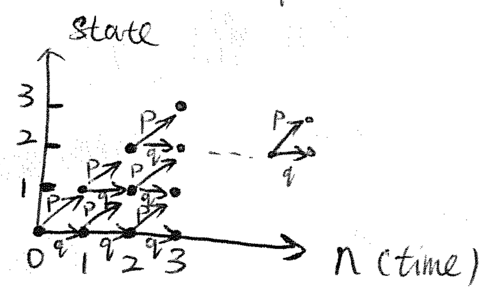
Diagrams:

1. State transition diagram (implicit time)

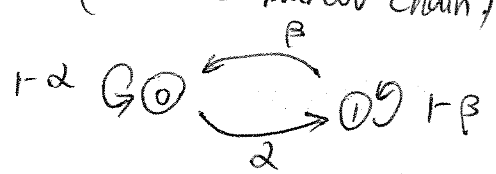
   E.g.1: (Binomial counting process)



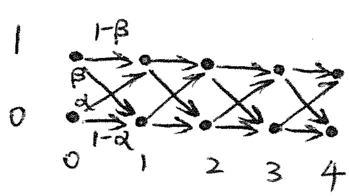
2. Trellis diagram (explicit time)



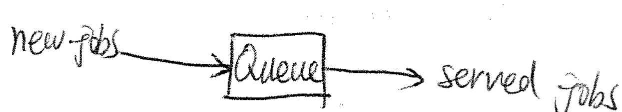
Eg. 2: (2-state Marcov chain)



(State transition)



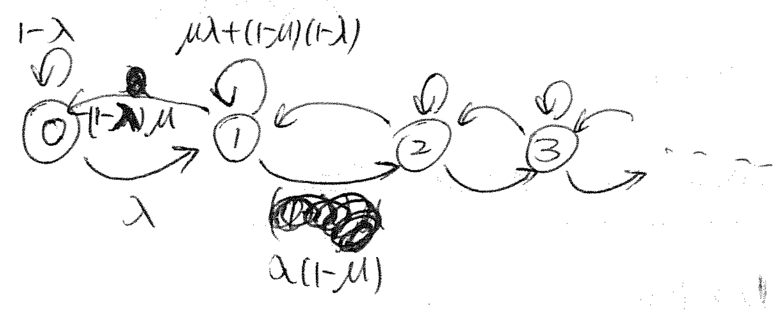
Eg. 3: (Queue).



X_n: # of jobs in queue.

new jobs arrive with prob. λ.

Job served w/ ~~pb~~ prob. μ. iid



<u>Def</u>: Discrete-time Markov chain is <u>homogeneous</u> if

$$P_{ij}(n) \equiv P_R\{X_{n+1}(u)=j \mid X_n(u)=i\} = P_{ij}$$

<u>Def</u>: <u>State transition probability matrix</u> is

$$P \equiv [P_{ij}]$$

with $P_{ij} \equiv P_R\{X_{n+1}(u)=j \mid X_n(u)=i\}$.

P is a (row) stochastic matrix:

$$\sum_j P_{ij} = 1.$$

E.g. 1: (Binomial counting process)

$$P = \begin{pmatrix} q & p & & 0 \\ & q & p & \\ & & q & \ddots \\ & 0 & & \ddots \end{pmatrix}_{\infty\times\infty}$$

E.g. 2: (2-state markov chain)

$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

Eg. 3: (Queue)

$$P = \begin{pmatrix} 1-\lambda & \lambda & & & 0 \\ (1-\lambda)\mu & \mu\lambda+(1-\mu)(1-\lambda) & \lambda(1-\mu) & & \\ & (1-\lambda)\mu & \mu\lambda+(1-\mu)(1-\lambda) & \ddots & \\ & 0 & \ddots & \ddots & \ddots \end{pmatrix}_{\infty\times\infty}$$

k-step state transition probabilities

Def: k-step state transition prob.

$$p_{ij}^{(k)} \equiv Pr\{X_{n+k}(u)=j \mid X_n(u)=i\}.$$

Thm: (Chapman-kolmogorov equation)

$$p_{ij}^{(k+l)} = \sum_m p_{im}^{(k)} p_{mj}^{(l)} \qquad \forall k, l \in \mathbb{Z}_{>0}, \ \forall i, j \in S.$$

Proof: $p_{ij}^{(k+l)} = Pr\{X_{n+k+l}(u)=j \mid X_n(u)=i\}$

$= \sum_m Pr\{X_{n+k+l}(u)=j \mid X_{n+k}(u)=m, X_n(u)=i\} \cdot Pr\{X_{n+k}(u)=m \mid X_n(u)=i\}$

$= \sum_m p_{im}^{(k)} p_{mj}^{(l)}$

State probability vector

Def: state probability vector is a row vector:

$\Pi(n) \equiv (\Pi_1(n), \Pi_2(n), \cdots)$

where $\Pi_i(n) \equiv Pr\{X_n(u)=i\}$.

And $\sum_i \Pi_i(n) = 1$.

Property:

$$\Pi_j(n+1) = Pr\{X_{n+1}(u) = j\}$$
$$= \sum_i Pr\{X_{n+1}(u) = j,\ X_n(u) = i\}$$
$$= \sum_i Pr\{X_{n+1}(u) = j \mid X_n(u) = i\}\, Pr\{X_n(u) = i\}$$
$$= \sum_i P_{ij}\, \Pi_i(n)$$

$$\Rightarrow \Pi(n+1) = \Pi(n) P. \qquad (n = 0, 1, 2, \cdots)$$

$$\Rightarrow \Pi(n) = \Pi(0) P^n \qquad (n = 0, 1, 2, \cdots)$$

~~n-step state transition prob. matrix~~

$$\Rightarrow P^n = P^{(n)}$$

Thm: (Matrix version of Chapman-kolmogorov equation)

$$P^{(k+l)} = P^{(k)} P^{(l)}$$

Proof: This is trivial since $P^{(n)} = P^n$.

Def: The steady state, or stationary distribution is

$$\Pi \equiv \lim_{n\to\infty} \Pi(n)$$

, when the limit exists, and is the same for any $\Pi(0)$.

Note that $\Pi = \Pi P$. (left Perron vector ?)

① Def: Reducible ~~discrete time Markov chain~~ DTMC

not all states can be reached from every state.

Note: If for any $i, j \in S$, state i can go to state j reducible DTMC can have steady state.

② **Def:** A state in a DTMC is <u>recurrent</u> if

$Pr\{\text{state } i \text{ is revisited} \mid \text{in state } i\} = 1$

If a state is not recurrent, we call it's <u>non-recurrent</u> or <u>transient</u>.

Note: 1. Recurrent state $\iff \sum_{m=1}^{\infty} P_{ii}^{(m)} \to \infty$

Transient state $\iff \sum_{m=1}^{\infty} P_{ii}^{(m)} < \infty$

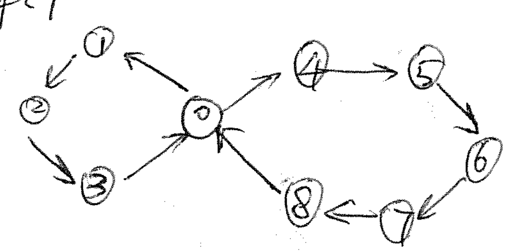
2. ~~An irreducible DTMC has either all recurrent states, or all transient states. (this can only happen for infinite states.)~~ ✓

Irreducible, finite state DTMC has all recurrent states.

③ **Def:** A state in a DTMC is <u>periodic</u> w/ period $d_i > 1$ if

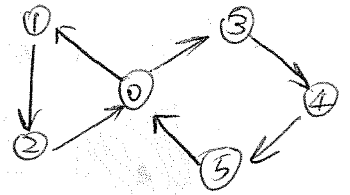
$P_{ii}^{(m)} = 0$, $\forall\, m \neq$ multiple of d_i.

Eg: 1°



state 0 is periodic:
$d_0 = \gcd(4,6) = 2$

2°



for state 0:
$\gcd(3,4) = 1$
is not periodic.

Note: 1. A DTMC that is irreducible has the same d_i for every state i.

①+②+③ <u>Def</u>: An irreducible, aperiodic, positive recurrent DTMC is called an <u>ergodic</u> DTMC.

<u>Def</u>: The time to reach state j for the first time given $X_0(u) = i$ is a random variable <u>$T_{ij}(u)$</u>, with $1 \le T_{ij}(u) < \infty$.

$$m_{ij} \equiv E\{T_{ij}(u)\}$$

and m_{ii} is called mean recurrence time.

Thm: $$m_{ij} = 1 + \sum_{v \neq j} P_{iv}\, m_{vj}$$

Proof: Let $P_{ij}^{*(T_{ij})}$ the prob. from state i reaching state j for the first time at time T_{ij}.

$$m_{ij} = \sum_{T_{ij}=1}^{\infty} T_{ij}\, P_{ij}^{*(T_{ij})}$$

$$= P_{ij} + \sum_{T_{ij}=2}^{\infty} T_{ij} P_{ij}^{*(T_{ij})}$$

$$= P_{ij} + \sum_{T_{ij}=2}^{\infty} T_{ij} \sum_{v \neq j} P_{iv} P_{vj}^{*(T_{ij}-1)}$$

$$= P_{ij} + \sum_{t=1}^{\infty} (t+1) \sum_{v \neq j} P_{iv} P_{vj}^{*(t)}$$

$$= P_{ij} + \sum_{v \neq j} P_{iv} \left(\sum_{t=1}^{\infty} t P_{vj}^{*(t)} + \sum_{t=1}^{\infty} P_{vj}^{*(t)} \right)$$

$$\because \sum_{t=1}^{\infty} p_{ij}^{*(t)} = 1 \quad \text{(irreducibility)}$$

$$\sum_{t=1}^{\infty} t\, p_{vj}^{*(t)} = m_{vj}$$

$$\therefore m_{ij} = p_{ij} + \sum_{v \neq j} p_{iv} (m_{vj} + 1)$$

$$= p_{ij} + \sum_{v \neq j} p_{iv} + \sum_{v \neq j} p_{iv} m_{vj}$$

$$= 1 + \sum_{v \neq j} p_{iv} m_{vj} . \quad \square .$$

Fact: For an irreducible DTMC either

① $m_{ij} < \infty$, $\forall ij \in \mathcal{S}$ $\Rightarrow$ all states are positive recurrent

② $m_{ii} = \infty$, $\forall i \in \mathcal{S}$ $\Rightarrow$ all states are null recurrent.

<u>Thm</u>: There are 2 cases for asymptotic behavior of an irreducible, aperiodic DTMC:

(A) the DTMC is not ergodic (null recurrent), then

$p_{ij}^{(n)} \to 0$, $\forall ij$, there is no stationary Π.

(B) the DTMC is ergodic (positive recurrent), then

- $\Pi(n) \to \Pi$, there is a unique solution to $\Pi = \Pi P$.
- $p_{ij}^{(n)} \to \Pi_j$, $\forall j \in \mathcal{S}$, regardless of i.
- $\Pi_j = \dfrac{1}{m_{jj}}$, $\forall j \in \mathcal{S}$
- (strong law of large numbers) Let $N_j(u,n)$ is the r.v. that is # of times state j visited in $\{0, 1, \dots, n\}$. $\lim \frac{N_j(u,n)}{} = \Pi_j$ [illegible]

Methods to find Π:

1. Cut-sets:

<u>Def</u>: Given a DTMC, a <u>cut</u> of $\mathcal{S}$ is a patitioning into a, B.

Cut-set equations: For an ergodic DTMC, the stationary distribution Π satisfies

$$\sum_{i \in a} \sum_{j \in B} \pi_i P_{ij} = \sum_{k \in B} \sum_{m \in a} \pi_k P_{km}, \quad \forall \text{ cuts } a, B.$$

2. Detailed balance equations:

<u>Thm</u>: For ergodic DTMC, if exists a probability vector α that satisfies

$$\alpha_i P_{ij} = \alpha_j P_{ji}, \quad \forall i, j \in \mathcal{S}$$

then $\Pi = \alpha$.

proof:
$$\alpha_j = \alpha_j \sum_i P_{ji}$$
$$= \sum_i \alpha_j P_{ji}$$
$$= \sum_i \alpha_i P_{ij}$$

$\therefore \alpha = \alpha P$

Since the DTMC is ergodic.

$\therefore \Pi = \alpha$.

Eg 1° B/B/1 Queue stationary distribution? $(\lambda < \mu)$

B / B / 1:
- Bernoulli arrival model
- Bernoulli service model
- 1 queue

$$\begin{cases} \pi_0 = \dfrac{1}{1 + \dfrac{\lambda}{\mu\bar{\lambda} - \lambda\bar{\mu}}} \\ \pi_i = \pi_0 \left(\dfrac{\lambda}{\mu\bar{\lambda}}\right) \cdot \left(\dfrac{\lambda\bar{\mu}}{\mu\bar{\lambda}}\right)^{i-1} \qquad (i = 1, 2, \cdots) \end{cases}$$

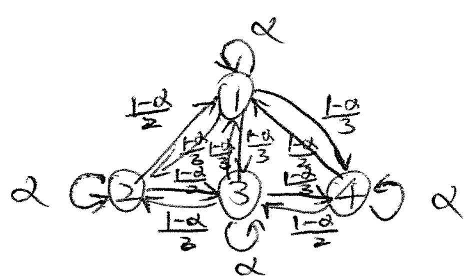
where $\bar{\lambda} = 1 - \lambda$, $\bar{\mu} = 1 - \mu$.

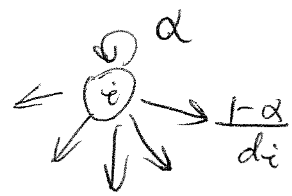
Eg 2°: (Random walk over a finite, connected, undirected graph)

$X_n(u) \in \{1, 2, \cdots, N\}$

$X_{n+1}(u) = X_n(u)$ occurs w/ prob. α.

$X_{n+1}(u)$ is a randomly selected neighbor with equal prob.





~~d_i = # of neigh~~

d_i: degree of node i in the original graph.

$$\pi = \frac{1}{\sum_{j=1}^{N} d_j} (d_1, d_2, \cdots, d_N)$$

Let $\alpha_i = C d_i$,

then
$$\begin{cases} \alpha_i P_{ij} = C d_i \dfrac{1-\alpha}{d_i} = C(1-\alpha) \\ \alpha_j P_{ji} = C d_j \dfrac{1-\alpha}{d_j} = C(1-\alpha) \end{cases} \quad \forall i j.$$

$\therefore \alpha_i P_{ij} = \alpha_j P_{ji}$, $\forall i j$.