IV. mDOF

① <u>Vibration of mDOF systems</u> (construction of equations)

$$k_{ij} = k_{ji} = \frac{\partial^2 U_{el}}{\partial x_i \, \partial x_j}$$

where k_{ij} is stiffness coefficient,
$[k]$ is stiffness matrix,
U_{el} is the strain energy.

$$\underset{\sim}{F_{el}} = -[k]\,\underset{\sim}{x}$$

where $\underset{\sim}{F_{el}}$ are the generalized elastic forces,
$\underset{\sim}{x}$ are the generalized displacements.

$$\underset{\sim}{F_{in}} = -[M]\,\ddot{\underset{\sim}{x}}$$

where $\underset{\sim}{F_{in}}$ are the generalized inertia forces,
$[M]$ is the mass matrix,
$\ddot{\underset{\sim}{x}}$ are the generalized accelerations.

$$T = \frac{1}{2}\,\dot{\underset{\sim}{x}}^T [m]\,\dot{\underset{\sim}{x}}$$

where T is the kinetic energy,
$\dot{\underset{\sim}{x}}$ are the generalized velocities.

$$\delta W = \delta \underset{\sim}{x} \cdot \underset{\sim}{F_{ext}}$$

where $\underset{\sim}{F_{ext}}$ is the generalized external force.

② Solution of mDOF systems

General solution for free vibration: (no damping ; modal analysis)

$$[M]\ddot{\underset{\sim}{x}} + [k]\underset{\sim}{x} = \underset{\sim}{0}$$

Assume normal mode solution:

$$\underset{\sim}{x}(t) = \underset{\sim}{u} f(t)$$

$$\Rightarrow [M]\underset{\sim}{u}\ddot{f}(t) + [k]\underset{\sim}{u}f(t) = \underset{\sim}{0}$$

$$\Rightarrow m_{ij}u_j\ddot{f}(t) + k_{ij}u_j f(t) = 0$$

$$\Rightarrow -\frac{\ddot{f}(t)}{f(t)} = \frac{k_{ij}u_j}{m_{ij}u_j} = \text{Constant}$$

If constant ≤ 0, total energy will blow up, which is not physical.
Hence, constant > 0, suppose constant $= \omega^2$

$$\Rightarrow \ddot{f}(t) + \omega^2 f(t) = 0 \quad \text{and.}$$

$$(k_{ij} - \omega^2 m_{ij})u_j = 0$$

$$\Rightarrow f(t) = c\sin\omega_r t + c'\cos\omega_r t = q_r(t) \;(r=1,2,\cdots,n) \qquad (1)$$

where ω_r^2 are solutions of $\det([k]-\omega_r^2[M]) = 0$,
and the normal modes are normalized vectors satisfying:

$$([k] - \omega_r^2[M])\,\underset{\sim}{u}^{(r)} = \underset{\sim}{0}$$

$\Rightarrow$ The r^{th} normal mode solution is

$$\underset{\sim}{x}(t) = \underset{\sim}{u}^{(r)} q_r(t)$$

$\Rightarrow$ General solution is

$$\underset{\sim}{x}(t) = (\underset{\sim}{u}^{(1)}, \cdots, \underset{\sim}{u}^{(n)}) \cdot (q_1(t), \cdots, q_n(t))^T \qquad (2)$$

3 Orthogonality of Normal modes (generalized coordinates $\rightarrow$ normal coordinates)

Since $[k]\,\underset{\sim}{u}^{(i)} = \omega_i^2\,[m]\,\underset{\sim}{u}^{(i)}$

$[k]\,\underset{\sim}{u}^{(j)} = \omega_j^2\,[m]\,\underset{\sim}{u}^{(j)}$

when $\omega_i \neq \omega_j$,

$$\underset{\sim}{u}^{(j)T}\,[m]\,\underset{\sim}{u}^{(i)} = 0$$

$$\underset{\sim}{u}^{(j)T}\,[k]\,\underset{\sim}{u}^{(i)} = 0$$

If normal modes are normalized according to

$$\underset{\sim}{u}^{(j)T}\,[m]\,\underset{\sim}{u}^{(i)} = \delta_{ij}$$

then

$$[u]^T[m][u] = I$$

$$[u]^T[k][u] = \mathrm{diag}\{\omega_1^2, \cdots, \omega_n^2\}$$

Thm: (**Expansion theorem**)

If $\underset{\sim}{u}^{(i)}$ are normalized normal modes, then

$$\forall \underset{\sim}{x}, \quad \underset{\sim}{x} = \sum_{i=1}^{n} c_i\,\underset{\sim}{u}^{(i)}$$

where $c_i = \underset{\sim}{u}^{(i)T}\,[m]\,\underset{\sim}{x}$

For $[m]\,\ddot{\underset{\sim}{x}} + [k]\,\underset{\sim}{x} = \underset{\sim}{F}(t)$,

using <u>normal coordinates</u> $\underset{\sim}{q}$: $\underset{\sim}{x} = [u]\,\underset{\sim}{q}$,

the equations of motion are decoupled:

$$\mathrm{diag}\{m_1, \cdots, m_n\}\,\ddot{\underset{\sim}{q}} + \mathrm{diag}\{k_1, \cdots, k_n\}\,\underset{\sim}{q} = \underset{\sim}{Q}(t)$$

④ <u>General Response of Linear mDOF system (modal analysis)</u>

$$\begin{cases} [m]\ddot{\underset{\sim}{x}} + [c]\dot{\underset{\sim}{x}} + [k]\underset{\sim}{x} = \underset{\sim}{F}(t) \\ \underset{\sim}{x}(0) = \underset{\sim}{x}_0 \\ \dot{\underset{\sim}{x}}(0) = \dot{\underset{\sim}{x}}_0 \end{cases}$$
(Assuming : $[C] = \alpha[m] + \beta[k]$)

Solve for $[m]\ddot{\underset{\sim}{x}} + [k]\underset{\sim}{x} = \underset{\sim}{0}$,

we get $\underset{\sim}{x}(t) = [u]\ \underset{\sim}{q}_0(t)$ (see ② "general solution for free vibration.")

Using modal transformation (normal coordinates),

$$\underset{\sim}{x} = [u]\underset{\sim}{q}$$

the system gets decoupled :

$$\begin{cases} M\ddot{\underset{\sim}{q}} + C\dot{\underset{\sim}{q}} + K\underset{\sim}{q} = \underset{\sim}{Q}(t) \\ \underset{\sim}{q}(0) = [u]^{-1}\underset{\sim}{x}_0 \\ \dot{\underset{\sim}{q}}(0) = [u]^{-1}\dot{\underset{\sim}{x}}_0 \end{cases}$$

where $M = [u]^T[m][u]$, $C = [u]^T[c][u]$, $K = [u]^T[k][u]$
are all diagonal matrices,

and $\underset{\sim}{Q}(t) = [u]^T\ \underset{\sim}{F}(t)$

The solutions are

$$q_i(t) = e^{-\zeta_i \omega_i t}\left[q_i(0)\cos\omega_{d_i}t + \frac{\dot{q}_i(0) + \zeta_i\omega_i q_i(0)}{\omega_{d_i}}\sin\omega_{d_i}t\right] + \int_0^t Q_i(\tau)\,h_i(t-\tau)\,d\tau$$

with $h_i(t) = \frac{1}{\omega_{d_i}} e^{-\zeta_i\omega_i t}\sin\omega_{d_i}t$, and ω_i, ω_{d_i}, ζ_i similar as 1DOF.