Lecture 2 (metric space)

metric space : (X, d) , a pair of an underlying set X, and metric d.

$d(x, A) \equiv \inf\{d(x,y) : y \in A\}$, $A \subseteq X$

$\text{diam}(A) \equiv \sup\{d(x,y) : x, y \in A\}$.

bounded set : $\text{diam}(A) < \infty$.

subspace : (A, d) , $A \subseteq X$.

product space : (Z, d_Z) , $Z = X \times Y$, $d_Z = d_X \times d_Y$; (X, d_X), (Y, d_Y)

$(d_Z(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2))$

metric space of continuous fns :

~~continuous~~ mapping : $F : X \to Y$, (X, d_1), (Y, d_2)

continuity

uniform continuity.

* functions as element

E.g. : metric space $(\mathcal{L}_2([0,T], \mathbb{R}), d_2)$

kernal fn $k(t, \tau) \in \mathcal{L}_2([0,T] \times [0,T], \mathbb{R})$

* operator / transforms as mapping

mapping $K : \mathcal{L}_2([0,T], \mathbb{R}) \to \mathcal{L}_2([0,T], \mathbb{R})$

Lecture 3

P69 #11 continuous mapping on product space.

A same sequence may converge in one metric, but does not in another.

Convergence in metric space.

Lecture 4

P_{73} #14. 1° $t_n \to t$ in $(\mathbb{R}, d)$

2° in $(C([0,T], \mathbb{R}), d_\infty)$, $f_n \to f$

then 3° $f_n(t_n) \to f(t)$ in $(\mathbb{R}, d)$

P_{75} #5

P_{76} #10 (b) $x_n(t) = \frac{1}{nt}$

$\to x(t) = 0$ in $(C([1, +\infty), \mathbb{R}), d_2)$

But $\lim_{n\to\infty} \int_I x_n \, dt \neq \int_I x \, dt = 0$

↑ doesn't exist.

Letture 5 (Open set)

* continuity and convergence are independent of the metric.
* Topology is generated by metric, but independent of it.

Local neighborhoods :

- **open ball** (of radius r) (centered at x_0) : $B_r(x_0) \equiv \{ x \in X : d(x, x_0) < r \}$ in (X, d) $(r > 0)$
- **closed ball** : $B_r[x_0] = \{ x \in X : d(x, x_0) \leq r \}$ in (X, d) $(r \geq 0)$
- **sphere** : $S_r[x_0] = \{ x \in X : d(x, x_0) = r \}$ in (X, d) $(r \geq 0)$

Lemma: $\forall x \in B_r(x_0)$, $\exists B_\rho(x)$, s.t. $B_\rho(x) \subseteq B_r(x_0)$

Thm: Function $F : (X, d_1) \to (Y, d_2)$ is continuous at x_0 $\Leftrightarrow$ $\forall r, \underset{\exists \rho, s.t.}{} F^{-1}(B_r(F(x_0))) \supseteq B_\rho(x_0)$

open set : Set $A \subseteq (X, d)$ is <u>open</u>, if $\forall x \in A$, $\exists B_r(x) \subseteq A$.

Topology : The topology $\mathcal{T}$ of set X, generated by metric d, is the class of all open sets in (X, d).

Thm: 1° $\emptyset, X \in \mathcal{T}$; 2° if $A_\alpha \in \mathcal{T}$, then $\bigcup_\alpha A_\alpha \in \mathcal{T}$; 3° if $A_1, \cdots, A_n \in \mathcal{T}$, then $\bigcap_{i=1}^{n} A_i \in \mathcal{T}$.

open mapping : Mapping $F : (X, d_1) \to (Y, d_2)$ is open, if open set A in (X, d_1) indicates open set $F(A)$ in (Y, d_2).

Homeomorphism: If fn $F : (X, d_1) \to (Y, d_2)$ is cont. & 1-1 onto Y & F^{-1} is cont., then F is a <u>homeomorphism</u>, and (X, d_1) and (Y, d_2) are <u>homeomorphic</u>.

Isometry: ~~Homeomorphism~~ Function F is an <u>isometry</u>, if $d_2(F(x_1), F(x_2)) = d_1(x_1, x_2)$, $\forall x_1, x_2 \in X$.

Note: Isometry is stronger than homeomorphism.

Equivalent metrics: For metric spaces (X, d_1), (X, d_2), TFAE:

(1) d_1, d_2 are <u>equivalent</u>

(2) $I : (X, d_1) \to (X, d_2)$ and $I^{-1} : (X, d_2) \to (X, d_1)$ are continuous

(3) $\forall (Y, d_3)$,

$F : (X, d_1) \to (Y, d_3)$ is continuous $\Leftrightarrow$ $F : (X, d_2) \to (Y, d_3)$ is continuous

(4) $x_n \to x_0$ in (X, d_1) $\Leftrightarrow$ $x_n \to x_0$ in (X, d_2).

Thm: d_1, d_2 are equivalent $\Leftrightarrow$ $\mathcal{T}_1 = \mathcal{T}_2$

Connected set: Set $A \subseteq (X,d)$ is <u>disconnected</u>, if

$\exists$ open sets $B_1, B_2 \subseteq (X,d)$, $B_1, B_2 \neq \emptyset$, $B_1 \cap B_2 = \emptyset$, s.t.

$A = B_1 \cup B_2$

A set is <u>connected</u>, if it is not disconnected.

Lecture 6 (Closed set).

Closed set: $A \subset X$ is $\underbrace{\text{closed}}_{\text{in } (X,d)}$, if A^c is open.

lemma: $B_r[x_0]$ and $S_r(x_0)$ $(r \geq 0)$ are closed sets in (X,d).

Thm: Denote $\mathcal{F}$ to be the collection of all closed sets in (X,d), then

(1) $\emptyset, X \in \mathcal{F}$

(2) $\bigcap_\alpha A_\alpha \in \mathcal{F}$

(3) $\bigcup_{i=1}^n A_i \in \mathcal{F}$.

Def: <u>point of accumulation (adherence)</u> of A: $A \subseteq (X,d)$, $x \in (X,d)$, $\exists \{a_n\} \subseteq A$, s.t. $a_n \xrightarrow{d} x$.

Def: Set $\bar{A}$ is the <u>closure</u> of A, if it is the set of all points of adherence in A.

Note: $\bar{A} \subseteq X$.

Def: set $A \subseteq (X,d)$ is <u>dense</u> in X, if $\bar{A} = X$.

Lecture 7 (Countability; real number)

Countable: $S \sim \mathbb{N}$, S has 1-1 correspondence with $\mathbb{N}$; otherwise, it's uncountable

Equivalence of sets: $M \sim N$, if $\exists$ 1-1 correspondence between M and N.

binary seq.: $\tau: \mathcal{B} \to [0,1] \in \mathbb{R}$, $\mathcal{B}$ is the class of binary sequences.

where $b \xrightarrow{\tau} \sum_{k=1}^{\infty} \frac{a_k}{2^k}$.

Note: 1° τ is not 1-1.

2° $\tau': \mathcal{B} \to [0,1)$ is 1-1, if require $\forall k \in \mathbb{N}$, a_k is the largest in $\{0,1\}$, s.t. $\sum_{i=1}^{k} \frac{a_i}{2^i} \leq r$.

$\mathbb{R}$ is uncountable.

$A^{\text{uncountable}} \setminus B^{\text{countable}} = C^{\text{uncountable}}$.

$\mathbb{Q}$ is dense in $\mathbb{R}$.

Def: The power of a set A is denoted as $m(A)$.

1° $m(A)=m(B)$ if $A \sim B$

2° $m(A)>m(B)$ if $\exists M \subset A$, s.t. $M \sim B$, but $\nexists N \subset B$, s.t. $A \sim N$.

3° $m(A)<m(B)$ is similarly defined.

Def: 1° If $A \sim \mathbb{N}$, then A is said to have the power of a countable set, denote $m(A) = \aleph_0$ (aleph null)

2° If $A \sim \mathbb{R}$, then A is said to have the power of the continuum, denote $m(A) = \aleph$.

Thm: The power set of M is $\mathcal{M}$, then $m(\mathcal{M}) > m(M)$.

Lecture 8 (separability)

Def: (X,d) is separable, if $\exists A \subset X$, A is dense and countable.

lemma: (X,d) is separable $\Leftrightarrow$ $\exists \{x_n\}$, $\forall \varepsilon>0$, $\forall x \in X$, $\exists x_n$, s.t. $d(x_n,x)<\varepsilon$.

Note: 1° (l_p, d_p) is separable, where $p \in [1,\infty)$, $l_p = \{x=(x_1,x_2,\cdots) \mid d_p(x,0)<\infty\}$, $d_p(x,y) = \left[\sum_{i=1}^{\infty} |x_i - y_i|^p\right]^{\frac{1}{p}}$.

2° (l_∞, d_∞) is not separable.

Lecture 9 (Completeness)

Def: (X,d) is complete, if every Cauchy seq. converges.

e.g.: (l_p, d_p) $(p \in [1,+\infty))$ is complete.

(C, d_2) is not complete.

Thm: (X,d) is complete $\Leftrightarrow$ $\exists a \in X$, s.t. $\forall$ decreasing non-empty closed sets $A_1 \supset A_2 \supset \cdots \supset A_m \supset \cdots$ with $\operatorname{diam} A_m \to 0$, $\bigcap_{m=1}^{\infty} A_m = \{a\}$.

Baire's Thm: Complete metric space (X,d) is the countable union of sets $A_1, A_2, \cdots$, then at lease one set A_n has nonempty interior.

Lect. 10

* Hierarchy of spaces:
  - 0° (X, d)
  - 1° $\{x\}$
  - 2° P
  - 3° (X^*, Δ)

(Equivalence class) (Completion)

Def: seq $\{p_n\}, \{q_n\}$, $p_n, q_n \in (X, d)$, are <u>equivalent</u> if

$$\lim_{n\to\infty} d(p_n, q_n) = 0.$$ Denote as $\{p_n\} \sim \{q_n\}$

Properties: 1° Reflective

2° Symmetric

3° transitive

Thm: (X^*, Δ) is a complete metric space. (of Cauchy equivalent classes.)

where $X^* = \{P \mid \forall \{p_n\}, \{p_n'\} \in P, \{p_n\} \sim \{p_n'\}$, and are all Cauchy seq.$\}$,

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n), \forall \{p_n\} \in P, \forall \{q_n\} \in Q.$$

Lemma 1°: $\Delta(P, Q)$ exists. (Hint: $|d(p_n, q_n) - d(p_m, q_m)| \underset{<\varepsilon}{\leq} d(p_n, p_m) + d(q_n, q_m)$)

Properties: 1° $\Delta(P, Q)$ is independent of the choice of $\{p_n\} \in P$, and $\{q_n\} \in Q$.

2° $\Delta(P, Q) = 0 \iff P = Q$

3° $\Delta(P, Q) = \Delta(Q, P)$

4° $\Delta(P, Q) \leq \Delta(P, R) + \Delta(R, Q)$

Needs proof → Lemma 2°: $\varphi(X)$ is dense in X^*, where $\varphi: X \to X^*$,

$\varphi(p) = P_p$, Where P_p is the (Cauchy) equivalent class which includes $\{p, p, \cdots\}$.

Thm: (X^*, Δ) is a completion of (X, d).

Lec-11 (Contraction Mapping)

Def: **Contraction** : $f:(X,d) \to (X,d)$ is a contraction mapping, if $\exists K \in [0,1)$, s.t. $d(f(x), f(y)) \le k\, d(x,y)$, $\forall x,y \in X$.

Thm: (Contraction mapping thm)

If (X,d) is complete and $f: X \to X$ is a contraction, then f has a unique fixed point and every sequence of iterations of f converges to this point.

Note: ~~Lyapunov fn. is a special case of contraction~~.
Contraction vs. Lyapunov fn.

Lec. 12 (Compactness)

Def.: Sequentially compact.

1° (X,d) : ~~[illegible]~~ every sequence has a convergent subsequen[ce]

2° $A \subseteq X$: subspace (A,d) is sequencially compact.

Note: A is "relatively compact", if $\bar{A}$ is compact.

Thm: 1° A seq. compact $\overset{(X,d)\text{ seq. compact}}{\Longleftrightarrow}$ A closed

2° (X,d) seq. compact $\Rightarrow (X,d)$ complete

Thm: (X,d) seq. compact, $\{M_n\}$ is a decreasing seq. of nonempty close sets, $\operatorname{diam} M_n \to 0$,

then $\bigcap_{m=1}^{\infty} M_m = \{a\}$

Def: Boltzano-Weierstras property: every infinite subset of (X,d) has at least one point of accumulation.

Def: Covering: $A \subset (X,d)$, $\forall \alpha$, $M_\alpha \subset (X,d)$, $\{M_\alpha\}$ is a covering of A if $A \subset \bigcup_\alpha M_\alpha$

Sub-covering: $\{M_\beta\}$ is a subcovering of $\{M_\alpha\}$, if $\{M_\beta\}$ is a subcollection of $\{M_\alpha\}$ that covers A.

open covering: A covering made up entirely of open sets.

Compact: Every open covering of (X,d) contains a finite open subcovering.

Thm: (Compactness Thm) $A \subseteq (X,d)$, TFAE:

(a) A is seq. compact

(b) A is compact

(c) A has the B-W property.

(d) A is $\mathcal{H}$-compact. (complete and totally bounded).

Thm: 1° $F: X \to Y$ cont., (X,d_1) compact, then $(F(X), d_2)$ compact.

2° $(X_1,d_1), \cdots, (X_m,d_m)$ compact, then $(X_1 \times \cdots \times X_m, d)$ is compact. ($d = \sum_i d_i$)

Note: Can be extended to product space of uncountably many spaces.

Thm: (B-W Thm)

Thm: ~~$f: (X,d) \to \mathbb{R}$~~

(X,d) is compact, $f: X \to \mathbb{R}$ is continuous, then f has maximum and minimum.

Def: (X,d_1) is compact, (Y,d_2) is complete, $G = G(X,Y)$ is the space of continuous fn. from X to Y. Metric $\rho(f,g) \equiv \sup\{d_2(f(x),g(x)) : x \in X\}$

1° $A \subset G(X,Y)$ is pointwise compact if $\forall x \in X$, $\{f(x) : f \in A\}$ has a compact closure in (Y,d_2)

2° $A \subset G(X,Y)$ is equi-continuous if $\forall x \in X$, $\forall \varepsilon > 0, \exists \delta > 0$, s.t.

~~$d_2(f(x), f(x')) \le \varepsilon$,~~

$\forall d_1(x_0,x') \le \delta$, $d_2(f(x),f(x')) \le \varepsilon$, $\forall f \in A$.

If δ is independent of x, then A is called uniformly equi-continuous.

Thm: (Arzela-Ascoli)

$A \subseteq (G,\rho)$, then

$\bar{A}$ is compact $\Leftrightarrow$ A is pointwise compact and equi-continuous.