Summary

$X(u,t) : \mathcal{U} \times \mathcal{T} \to \mathbb{C}$

For finite $\mathcal{T}$, $\mathcal{T} = \{1, 2, \cdots, n\}$,

$\underline{X}(u) = (X(u,1), \cdots, X(u,n))$ random vector.

Second moment description:

$\{ m_X(t), \; k_X(t_1,t_2), \; k_{XX}(t_1,t_2) \}$

$m_X(t) = \mathbb{E}\, X(u,t)$

$k_X(t_1,t_2) = \mathbb{E}[X(u,t_1) \overset{*}{X}(u,t_2)]$ (k_{XX^*})

$k_{XX}(t_1,t_2) = \mathbb{E}[X(u,t_1) X(u,t_2)]$

$R_X(t_1,t_2) = \mathbb{E}[X(u,t_1) \overset{*}{X}(u,t_2)]$ (R_{XX^*})

(Finite version)

$\{\underline{m}_X, k_X\}$

$\underline{m}_X = \mathbb{E}\underline{X}$; $k_X = \mathbb{E}(\underline{X}_0 \underline{X}_0^*)$ (k_{XX^*})

$k_{XX} = \mathbb{E}(\underline{X}_0 \underline{X}_0^T)$

$R_X = \mathbb{E}(\underline{X}\underline{X}^*)$ (R_{XX^*})

Note: $R_X = k_X + \underline{m}_X \underline{m}_X^*$

k_X, R_X — positive semi-definite.

Estimation: Simulation

Write $k_X = H H^*$

1° Hermitian

write $k_X = U \Lambda U^*$,

then $H = U \Lambda^{\frac{1}{2}} U^*$.

2° Causal

write $k_X = LL^T$ (Cholesky decomposition)

then $H = L$.

Simulation $\underline{X}_0 = H\underline{W}$

Whitening $\underline{W} = H^{-1}\underline{X}_0$

$\underline{Y} = H\underline{X}$ gives

$$\begin{cases} \underline{m}_Y = H\underline{m}_X \\ k_Y = H k_X H^* \\ R_Y = H R_X H^* \end{cases}$$

Mean square projection:

1° $\lambda_n \le \mathbb{E}|\underline{b}_\theta^* \underline{X}|^2 \le \lambda_1$

where λ_i are eigenvalues of R_X.

2° $(\mathbb{E}|\underline{b}_\theta^* \underline{X}|^2, \theta)$ forms a "peanut diagram" on polar coordinate.

Mean square length:

$\mathbb{E}|\underline{X}|^2 = \mathbb{E}\{tr(\underline{X}\underline{X}^*)\} = tr(R_X)$

K-L expansion:

$\underline{X}_0 = \sum_{i=1}^{n} V_i \underline{e}_i = E\underline{V}$

With $k_Z = E \Lambda E^*$, $E = (\underline{e}_1, \cdots, \underline{e}_n)$

$k_V = \Lambda$, i.e. V_i are mean-zero, variance λ uncorrelated r.v.'s.

$$\begin{cases} \underline{m}_{Z|X} = k_{ZX^*} k_X^{-1} (\underline{X} - \underline{m}_X) + \underline{m}_Z \\ k_{Z|X} = k_Z - k_{ZX^*} k_X^{-1} k_{XZ^*}. \end{cases}$$

<u>Hypothesis test</u>

Observation $\underline{X}$ is affected by $\underline{N}$,
decide whether $\underline{s}_1$ or $\underline{s}_2$ is the signal.

$$H_1: \underline{X}(u) = \underline{s}_1 + \underline{N}(u)$$

$$H_2: \underline{X}(u) = \underline{s}_2 + \underline{N}(u)$$

Minimum distance decision:

$$|\underline{X} - \underline{s}_1| \underset{H_2}{\overset{H_1}{\lessgtr}} |\underline{X} - \underline{s}_2|$$

1° $\underline{N} = \underline{W}$ is a white noise: $\underline{m}_W = 0$, $k_W = I$

Decision rule:

$$\underline{X}^*(\underline{s}_1 - \underline{s}_2) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{|\underline{s}_1|^2 - |\underline{s}_2|^2}{2}$$

Chebychev bound for error probability:

$$P\{\text{error} \mid H_1 \text{ true}\} \le \frac{4}{|\underline{s}_1 - \underline{s}_2|^2}$$

same bound holds for type-II error.

Suppose $P(H_1 \text{ true}) = P(H_2 \text{ true}) = \frac{1}{2}$,
then $P(\text{error}) \le \frac{4}{|\underline{s}_1 - \underline{s}_2|^2}$.

2° $\underline{m}_N = 0$, k_N nonsingular

Whitening $\underline{W} = H^{-1}\underline{N}$ gives

$$H_1: H^{-1}\underline{X}(u) = H^{-1}\underline{s}_1 + \underline{W}$$

$$H_2: H^{-1}\underline{X}(u) = H^{-1}\underline{s}_2 + \underline{W}$$

Decision rule:

$$\underline{X}^* k_N^{-1}(\underline{s}_1 - \underline{s}_2) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{1}{2}(\underline{s}_1^* k_N^{-1} \underline{s}_1 - \underline{s}_2^* k_N^{-1} \underline{s}_2)$$

Chebychev bound for error probability:

$$P\{\text{error} \mid H_1 \text{ true}\} \le \frac{4}{(\underline{s}_1 - \underline{s}_2)^* k_N^{-1} (\underline{s}_1 - \underline{s}_2)}$$

Same bound for H_2 and general error.

3° $\begin{cases} \underline{m}_N = 0, \ k_N \text{ singular}, \\ \forall \text{ zero eigenvector } \underline{e}, \ \underline{e}^*(\underline{s}_1 - \underline{s}_2) = 0 \end{cases}$

K-L expansion: $\underline{N} = \sum_{j=1}^{m} M_j \underline{e}_j = E_m \underline{M}$

Dimension reduction: $\underline{M} = E_m^* \underline{N}$ gives

$$H_1: E_m^* \underline{X}(u) = E_m^* \underline{s}_1 + \underline{M}$$

$$H_2: E_m^* \underline{X}(u) = E_m^* \underline{s}_2 + \underline{M}.$$

reduces to 2°.

<u>LMMSE</u>

Given 2nd moment description of $\underline{X}^{(n)}$ and $\underline{Z}^{(l)}$.
Find $G \in \mathbb{R}^{l \times m}$ (and $b \in \mathbb{R}^l$), that minimizes
$E|\underline{Z} - \hat{\underline{Z}}|^2$, with $\hat{\underline{Z}} = G\underline{X}$ (or $G\underline{X} + \underline{b}$)

1° R_X nonsingular	k_X nonsingular
$G = R_{ZX} R_X^{-1}$	$G = k_{ZX^*} k_X^{-1}$
	$\underline{b} = -G\underline{m}_X + \underline{m}_Z$
$MSE = tr\{R_Z - R_{ZX^*} R_X^{-1} R_{XZ^*}\}$	$MSE = tr\{k_Z - k_{ZX^*} k_X^{-1} k_{XZ^*}\}$

2° R_X singular

$$R_X = \tilde{E}_m \Lambda_m \tilde{E}_m^*, \quad \begin{cases} \Lambda_m \in \mathbb{R}^{m \times m} \\ \tilde{E}_m \in \mathbb{R}^{l \times m} \end{cases}$$

$$\begin{cases} \underline{Y} = \tilde{E}_m^* \underline{X} \\ \hat{\underline{Z}} = R_{ZY^*} R_Y^{-1} \underline{Y} \end{cases}$$

$$\hat{\underline{Z}} = R_{ZY} \tilde{E}_m \tilde{\Lambda}_m^{-1} \tilde{E}_m^* \underline{X}.$$

3° Causal

$$\underline{W} = L^{-1}\underline{X}$$
$$\hat{\underline{Z}} = \mathbb{C}(R_{ZW^*})\underline{X} \quad \Rightarrow \quad \hat{\underline{Z}} = \mathbb{C}(R_{ZX^*} L^{-*}) L^{-1} \underline{X}$$

Summary.

Random processes $X(u,t) : \mathcal{U} \times \mathcal{T} \to \mathbb{C}$

Types of $\mathcal{T}$: 1. finite $\mathcal{T} = \{1, 2, \cdots, n\}$, $\underline{X}(u) = (X(u,1), \cdots, X(u,n))$ is a random vector.

2. discrete time $\mathcal{T} = \mathbb{Z}_+$, $\mathbb{Z}$.

3. continuous time $\mathcal{T} = \mathbb{R}$, $[0, T)$

Second moment description: ~~$\{m_X(t), R_X(t_1, t_2)\}$~~

~~$m_X(t) \equiv E X(u,t)$~~

~~$R_X(t) \equiv E(X(u,t_1) X^*(u,t_2))$~~

1. $\{m_X(t), k_X(t_1, t_2), k_{XX}(t_1, t_2)$

$m_X(t) \equiv E X(u,t)$

$k_X(t_1, t_2) \equiv E(X_0(u,t_1) X_0^*(u,t_2))$ (k_{XX^*})

$k_{XX}(t_1, t_2) \equiv E(X_0(u,t_1) X_0(u,t_2))$

$R_X(t_1, t_2) = E(X(u,t_1) X^*(u,t_2))$ (R_{XX^*})

Note: 1° $R_X(t_1,t_2) = k_X(t_1, t_2) + m_X(t_1) m_X^*(t_2)$

$k_X(t_1, t_2)$ — positive semi-definite..

2° If $X(u,t)$ is real, there's no difference between $k_X(t_1,t_2)$ and $k_{XX}(t_1,t_2)$.

2. Finite version $\mathcal{T} = \{1, 2, \cdots, n\}$

$\{\underline{m}_X, k_X\}$

$\underline{m}_X \equiv E\underline{X}$; $\underset{(k_{XX^*})}{k_X} \equiv E(\underline{X}_0 \underline{X}_0^*)$, $k_{XX} \equiv E(\underline{X}_0 \underline{X}_0^T)$

$\underset{(R_{XX^*})}{R_X} \equiv E(\underline{X} \underline{X}^*)$

R_X — positive semi-definite.

$\underline{Y} = H\underline{X}$

$$\begin{cases} \underline{m}_Y = H\,\underline{m}_X \\ R_Y = H R_X H^* \\ k_Y = H k_X H^* \end{cases}$$

Simulation

$\underline{X}_0 = H\,\underline{W}$

with $k_X = HH^*$

Whitening

$\underline{W} = H^{-1}\underline{X}_0$

1° Hermitian, write $k_X = U\Lambda U^*$

then $H = U\Lambda^{\frac{1}{2}}U^*$

2° Causal, $k_X = LL^T$ (Cholesky Decomposition).

then $H = L$.

~~3° other,~~ $\cancel{k_X = U\Lambda U^*}$

~~choose~~ $\cancel{H = U\Lambda^{\frac{1}{2}}}$

Mean square projection:

$$\lambda_n \le \mathbb{E}\,|\underline{b}_\theta^* \underline{X}|^2 \le \lambda_1 \quad , \quad \lambda_i \text{ are ordered eigenvalues of } R_X.$$

$\left(\mathbb{E}\,|\underline{b}_\theta^* \underline{X}|^2, \theta\right)$ forms a "peanut diagram" on polar coordinates.

Mean square length:

$$\mathbb{E}\,|\underline{X}|^2 = \mathbb{E}\left[tr(\underline{X}\,\underline{X}^*)\right] = tr(R_X)$$

K-L expansion:

$$\underline{Z}_0 = \sum_{i=1}^{n} V_i\,\underline{e}_i = E\,\underline{V}$$

with $k_Z = E\Lambda E^*$, $E = (\underline{e}_1, \cdots, \underline{e}_n)$

$k_V = \Lambda$, i.e. V_i are mean-zero, variance λ_i, uncorrelate r.v.'s

Hypothesis Test

Observation $\underline{X}$ is affected by noise $\underline{N}$, decide whether $\underline{s}_1$ or $\underline{s}_2$ is the true signal.

$$H_1: \quad \underline{X}(u) = \underline{s}_1 + \underline{N}(u)$$

$$H_2: \quad \underline{X}(u) = \underline{s}_2 + \underline{N}(u)$$

Minimum distance decision:

1° $\underline{N} = \underline{W}$ is a white noise:

$$\underline{m}_W = \underline{0} \; ; \quad K_W = I$$

$$|\underline{X} - \underline{s}_1| \overset{H_1}{\underset{H_2}{\lessgtr}} |\underline{X} - \underline{s}_2|$$

(Decision rule)

$$\Leftrightarrow \quad \underline{X}^*(\underline{s}_1 - \underline{s}_2) \overset{H_1}{\underset{H_2}{\gtrless}} \frac{|\underline{s}_1|^2 - |\underline{s}_2|^2}{2}$$

Chebychev bound for error probability:

$$P\{\text{error} \mid H_1 \text{ true}\} = P\left\{(\underline{s}_1 + \underline{W})^*(\underline{s}_1 - \underline{s}_2) \le \frac{|\underline{s}_1|^2 - |\underline{s}_2|^2}{2}\right\}$$

$$= P\left\{\underline{W}^*(\underline{s}_1 - \underline{s}_2) \le -\frac{|\underline{s}_1 - \underline{s}_2|^2}{2}\right\}$$

$$\le P\left\{|\underline{W}^*(\underline{s}_1 - \underline{s}_2)| \ge \frac{|\underline{s}_1 - \underline{s}_2|^2}{2}\right\}$$

$$\le \frac{(\underline{s}_1 - \underline{s}_2)^* R_W (\underline{s}_1 - \underline{s}_2)}{\frac{|\underline{s}_1 - \underline{s}_2|^4}{4}}$$

$$= \frac{4}{|\underline{s}_1 - \underline{s}_2|^2}$$

Similarly $P\{\text{error} \mid H_2 \text{ true}\} \le \frac{4}{|\underline{s}_1 - \underline{s}_2|^2}$

Suppose $P(H_1 \text{ true}) = P(H_2 \text{ true}) = \frac{1}{2}$, then

$$P(\text{error}) = \sum_{i=1,2} P\{\text{error} \mid H_i \text{ true}\}\, P(H_i \text{ true}) \le \frac{4}{|\underline{s}_1 - \underline{s}_2|}$$

2° $\underline{m}_N = \underline{0}$, K_N nonsingular.

Whitening transformation $\underline{W} = H^{-1}\underline{N}$ gives,

$$H_1 : \quad H^{-1}\underline{X}(u) = H^{-1}\underline{s}_1 + \underline{W}(u)$$

$$H_2 : \quad H^{-1}\underline{X}(u) = H^{-1}\underline{s}_2 + \underline{W}(u)$$

Using result in 1°, the decision rule :

$$\underline{X}^* K_N^{-1}(\underline{s}_1 - \underline{s}_2) \underset{H_2}{\overset{H_1}{\gtrless}} \frac{1}{2}\left(\underline{s}_1^* K_N^{-1}\underline{s}_1 - \underline{s}_2^* K_N^{-1}\underline{s}_2\right)$$

Chebychev bound for error probability :

$$P\{\text{error} \mid H_1 \text{ true}\} \leq \frac{4}{(\underline{s}_1 - \underline{s}_2)^* K_N^{-1}(\underline{s}_1 - \underline{s}_2)}$$

The same bound holds for H_2 and error in general.

3° ($\underline{m}_N = \underline{0}$, K_N singular , $\forall$ zero eigen-vector $\underline{e}$, $\underline{e}^*(\underline{s}_1 - \underline{s}_2) = 0$)

K-L expansion : $\underline{N} = \sum_{j=1}^{m} M_j \underline{e}_j = E_m \underline{M}$

where M_j are mean-zero, variance λ_j ($\lambda_j \neq 0$) uncorrelated r.v.'s, and $E_m = (\underline{e}_1, \cdots, \underline{e}_m)$.

Dimension reduction $\underline{M} = E_m^* \underline{N}$ gives,

$$H_1 : \quad E_m^* \underline{X}(u) = E_m^* \underline{s}_1 + \underline{M}$$

$$H_2 : \quad E_m^* \underline{X}(u) = E_m^* \underline{s}_2 + \underline{M}$$

This reduces to 2°.

Perfect decision :

1° $\underline{m}_N = \underline{0}$, k_N singular , $\exists$ zero eigen-vector $\underline{e}$, s.t. $\underline{e}^*(\underline{s}_1 - \underline{s}_2) \neq 0$

Decision rule :

$$H_1 \text{ true} \iff \underline{e}^* \underline{X} = \underline{e}^* \underline{s}_1$$

$$H_2 \text{ true} \iff \underline{e}^* \underline{X} = \underline{e}^* \underline{s}_2$$

2° $|\underline{N}| < \left| \frac{\underline{s}_1 - \underline{s}_2}{2} \right|$

Decision rule : (minimum ~~decision~~ distance decision)

$$|\underline{X} - \underline{s}_1| \underset{H_2}{\overset{H_1}{\lessgtr}} |\underline{X} - \underline{s}_2|$$

LMMSE (and AMMSE)

Given 2nd moment description of $\underline{X}$ (n-dim) and $\underline{Z}$ (l-dim) ,

Find $G \in R^{l \times m}$ (and $\underline{b} \in R^l$) , that minimizes

$$E|\underline{Z} - \hat{\underline{Z}}|^2 \quad \text{with} \quad \hat{\underline{Z}} \equiv G\underline{X} \quad (\text{or } G\underline{X} + \underline{b})$$

1° R_X nonsingular (k_X nonsingular)

$$G = R_{ZX^*} R_X^{-1} \qquad \left(\begin{aligned} G &= k_{ZX^*} k_X^{-1} \\ \underline{b} &= -G\underline{m}_X + \underline{m}_Z \end{aligned} \right)$$

$$MSE = tr\{R_Z - R_{ZX^*} R_X^{-1} R_{XZ^*}\} \quad \left(MSE = tr\{k_Z - k_{ZX^*} k_X^{-1} k_{XZ^*}\} \right)$$

2° R_X singular (k_X singular)

$R_X = \tilde{E}_m \tilde{\Lambda}_m \tilde{E}_m^*$, $\tilde{\Lambda}_m \in R^{m \times m}$, $\tilde{E}_m \in R^{l \times m}$; $k_X = E_m \Lambda_m E_m^*$, $\Lambda_m \in R^{m \times m}$, $E_m \in R^{l \times m}$.

$$\left\{ \begin{aligned} \underline{Y} &= \tilde{E}_m^* \underline{X} \\ \hat{\underline{Z}} &= R_{ZY^*} R_Y^{-1} \underline{Y} \end{aligned} \right. \qquad \left(\left\{ \begin{aligned} \underline{Y} &= E_m^* (\underline{X} - \underline{m}_X) \\ \hat{\underline{Z}} &= k_{ZY^*} k_Y^{-1} \underline{Y} + \underline{m}_Z \end{aligned} \right. \right)$$

$$\hat{\underline{Z}} = R_{ZY^*} \tilde{E}_m \tilde{\Lambda}_m^{-1} \tilde{E}_m^* \underline{X} \qquad \left(\hat{\underline{Z}} = k_{ZX^*} E_m \Lambda_m^{-1} E_m^* (\underline{X} - \underline{m}_X) + \underline{m}_Z \right)$$

3° Causal LMMSE

$$\begin{cases} \underline{W} = L^{-1} \underline{X} \\ \hat{\underline{Z}} = \mathbb{C}(R_{ZW^*}) \, \underline{W} \end{cases}$$

$$\hat{\underline{Z}} = \mathbb{C}(R_{ZX^*} L^{-*}) L^{-1} \underline{X}$$

AMMSE and MMSE estimator for Gaussian random vector.

$\underline{X}$ and $\underline{Z}$ are jointly Gaussian, with nonsingular k, then

$\underline{Z} \mid \underline{X} = \underline{x} \sim N(\underline{m}_{Z|X}, k_{Z|X})$

with

$$\underline{m}_{Z|X} = k_{ZX^*} k_X^{-1} (\underline{x} - \underline{m}_X) + \underline{m}_Z$$

$$k_{Z|X} = k_Z - k_{ZX^*} k_X^{-1} k_{XZ^*}$$

$\therefore \hat{Z}_{MMSE} = \hat{Z}_{AMMSE}$, with $MSE = tr(k_{Z|X})$

LTI Operators

$\mathcal{T}$	$\mathcal{L}$	$\oplus$	$\mathcal{F}$	$\mathcal{E}_f \in \mathcal{L}$?
$\mathbb{Z}_n \equiv \{0, \cdots, n-1\}$	$\mathbb{C}^n$	addition mod n	$\{\frac{\nu}{n} : \nu \in \mathbb{Z}_n\}$	✓
$\mathbb{Z}$	$l^2(\mathbb{Z})$	addition	$[-\frac{1}{2}, \frac{1}{2})$	X
$\mathbb{R}_T \equiv [0, T)$	$l^2[0,T)$	addition mod T	$\{\frac{\nu}{T} : \nu \in \mathbb{Z}\}$	✓
$\mathbb{R}$	$l^2(\mathbb{R})$	addition	$\mathbb{R}$	X

LTI Operators H on $\mathbb{C}^n$ are circulant matrices,

$$H = (h_{ij})_{n\times n} \quad \text{with } h_{ij} \text{ only depend on } \left((i-j) \bmod n\right).$$

$$H = E \Lambda E^*, \quad \text{with } E = [\tilde{e}_f] = [e^{i2\pi f t}]$$

$$f \in \{0, \tfrac{1}{n}, \cdots, \tfrac{n-1}{n}\},$$

$$t \in \{0, \cdots, n-1\}.$$

$$\Lambda = \mathrm{diag}\{\lambda_f\}$$

$$\lambda_f = (h_{00}, \cdots, h_{0(n-1)})\, \tilde{e}_f$$

$$E = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{i\cdot 2\pi\cdot\frac{1}{n}} & \cdots & e^{i\cdot 2\pi\cdot\frac{n-1}{n}} \\ 1 & e^{i\cdot 2\pi\cdot\frac{1}{n}\cdot 2} & & \vdots \\ \vdots & \vdots & & \vdots \\ 1 & e^{i\cdot 2\pi\cdot\frac{1}{n}\cdot(n-1)} & \cdots & e^{i\cdot 2\pi\cdot\frac{n-1}{n}\cdot(n-1)} \end{pmatrix}$$

$$\begin{matrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \\ 4 & 8 & 12 \end{matrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{i\cdot\frac{2\pi}{n}\cdot 1} & e^{i\cdot\frac{2\pi}{n}\cdot 2} & \cdots & e^{i\cdot\frac{2\pi}{n}(n-1)} \\ 1 & e^{i\cdot\frac{2\pi}{n}\cdot 2} & e^{i\cdot\frac{2\pi}{n}\cdot 4} & \cdots & e^{i\cdot\frac{2\pi}{n}\cdot 2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & e^{i\cdot\frac{2\pi}{n}(n-1)} & e^{i\cdot\frac{2\pi}{n}\cdot 2(n-1)} & \cdots & e^{i\cdot\frac{2\pi}{n}\cdot(n-1)(n-1)} \end{pmatrix}$$