Nov. 5

Remark (Rmk): Optimal strategy does not always lead to constant payoff.

~~Value of game~~

~~Payoff matrix~~

General setup (Two-player zero-sum game):

P_1 has n possible options;

P_2 ... m — — — ;

~~Payoff~~

Payoff matrix $A_{m \times n}$: $A_{i,j}$ = payment P_1 gets from P_2, when P_1 does jth option, P_2 does ith option.

zero sum : Gain of P_1 = Loss of P_2

Note: Symmetric matrices lead to biased games (value of game $\neq 0$)

P_1 strategy : $x = [x_1, \cdots, x_n]^T$, $(x_i \geq 0, \sum_i x_i = 1)$

P_2 strategy : $y = [y_1, \cdots, y_n]$, $(y_i \geq 0, \sum_i y_i = 1)$

Expected payoff (value of game):

$$\sum_{i,j} A_{ij} x_j y_i = yAx.$$

Prop: If A skew symmetric, then the game is fair

General soln:

reduce a 2-variable optimization prob. to two sequential 1-variable optimization.

$$P_1 : x \longrightarrow P_2 : \hat{y} \Big| \min_y yAx \longrightarrow P_1 : \hat{x} \Big| \max_x \min_y yAx \qquad (*)$$

$$P_2 : y \longrightarrow P_1 : x \Big| \max_x yAx \longrightarrow P_2 : \hat{y} \Big| \min_y \max_x yAx \qquad (**)$$

Thm: (von Neumann)

$\forall A_{m \times n}$, minimax over all strategies equals the maximin:

$$(*) = (**)$$

"saddle point":

$$\hat{y}Ax \le \hat{y}A\hat{x} \le yA\hat{x} \quad , \forall x, y.$$

Proof: $\max_x \min_y yAx = \min_y yA\hat{x} \le \hat{y}A\hat{x} \le \max_x \hat{y}Ax = \min_y \max_x yAx$

Note: P1 and P2 have "duel" roles, both considering strategies from the "feasible set" of probability vectors.

Primal	Dual
$\min cx$	$\max yb$
$Ax \ge b,\ x \ge 0$	$yA \le c,\ y \ge 0$

Thm: (Duality) Linear programming guarantees $\exists$ feasible $\hat{x}, \hat{y}$, s.t. $c\hat{x} = \hat{y}b$.

$$A\hat{x} \ge b \Rightarrow yA\hat{x} \ge yb = 1$$

$$\hat{y}A \le c \Rightarrow \hat{y}Ax \le cx = 1$$

$$\hat{y}Ax \le 1 \le yA\hat{x}$$

$$\max_x \hat{y}Ax \le 1 \le \min_y yA\hat{x}$$