≪ Notes For Petroleum Reservoir Simulation ≫

Darcy's Law : Incompressible (Anisotropic)

$$
① \begin{cases} \vec{q} = -\dfrac{\bar{k}}{\mu}\rho \cdot \nabla \phi \\ \phi = gz + \displaystyle\int_{p_0}^{p} \dfrac{dp}{\rho} \end{cases}
$$

$\vec{q}$: local filter velocity

$$
② \quad \vec{q} = -\frac{\bar{k}}{\mu}\cdot(\nabla p - \rho\vec{g})
$$

Chap3. General Single Phase Simulator :

(Single phase, single component ; fluid and rock slightly compressible ; 3D, gravity included, oil field unit)

$$
(m_{i-\frac{1}{2},j,k} - m_{i+\frac{1}{2},j,k} + m_{i,j-\frac{1}{2},k} - m_{i,j+\frac{1}{2},k} + m_{i,j,k-\frac{1}{2}} - m_{i,j,k+\frac{1}{2}}) - m^{w}_{i,j,k} = \frac{1}{\Delta t}(M^{n+1} - M^{n})_{i,j,k}
$$

1° Mass Flux :

$$
\begin{cases} (x \text{ and } y) & m_{i-\frac{1}{2},j,k} = \rho^{o}(\Upsilon_x)_{i-\frac{1}{2},j,k}(p_{i-1,j,k} - p_{i,j,k}) \\ (z) & m_{i,j,k-\frac{1}{2}} = \rho^{o}(\Upsilon_z)_{i,j,k-\frac{1}{2}}(p_{i,j,k-1} - p_{i,j,k} + \gamma_{i,j,k-\frac{1}{2}}\Delta Z_{i,j,k} \end{cases}
$$

① transmissibility : $(\Upsilon_x)_{i-\frac{1}{2},j,k} = \alpha\left(\dfrac{\Delta y \Delta z\, k_x}{B(p)\, \mu\, \Delta x}\right)_{i-\frac{1}{2},j,k}$

~~单位转换参数~~:

$$
\begin{cases} \alpha = 0.001127 \ \dfrac{bbl \cdot cP}{ft \cdot day \cdot md \cdot psi} \\ \beta = \dfrac{1}{144}\dfrac{ft^2}{in^2} \\ \gamma = \beta\rho\dfrac{g}{g_c} \\ g_c = 32.2 \dfrac{lb_m}{lb_f}(ft \cdot s^{-2}) \end{cases}
$$

B : oil formation volume factor.

For this case, $B(p) = \dfrac{\rho^{o}}{\rho(p)}$, $\rho^{o} = \rho_{STC}$

2° Well Term : $m^{w}_{i,j,k} = \rho^{o} q^{w}_{i,j,k}$ (note : q^{w} in STC)

3° Mass Accumulation :

$$
M^{n+1}_{i,j,k} - M^{n}_{i,j,k} = \frac{\rho^{o}}{5.615}(\Delta x \Delta y \Delta z)_{i,j,k}\left(\phi^{n+1}C_f + \frac{\phi^{o}}{B^{n}}C_R\right)_{i,j,k}(p^{n+1}_{i,j,k} - p^{n}_{i,j,k})
$$

C_f : isothermal fluid compressibility

C_R : rock compressibility

5.615是 $ft^3 \to STB$ 的转换系数.

Note：1° 这不是一个完整的格式。基本未知量为 p，剩余未知量有 $\rho(p)$，$\bar{k}$，q^w，ϕ，以及常数 C_f，C_R.

待补充两个方程：$\rho(p)$本构关系；$\bar{k}$或ϕ与p的关系。

2° 边条件可以为 $m=0$

3° 井项 q^w 须另作考虑。

Oil - water system :

Assumptions : 1° two component, two phase (oil, water)

2° the phases are completely immiscible

3° 1D

Notation : subscript $l = o, w$

Governing Equations: 1° $\frac{\partial}{\partial x}\left[k \frac{\lambda_l}{B_l}\left(\frac{\partial p_l}{\partial x} - \gamma_l \frac{\partial z}{\partial x}\right)\right] = \frac{\partial}{\partial t}\left(\phi \frac{S_l}{B_l}\right) + \tilde{q}_l$

$\left(\lambda_l = \frac{k_{rl}}{\mu_l}\right)$

Unknowns : (p_o, p_w, S_o, S_w)

2° $P_c(S_w) = p_o - p_w$

3° $S_o + S_w = 1$

Primary Variables: (p_o, S_w)

1° Oil conservation equation

In a cell, $\int_{V_i} \frac{\partial}{\partial x}\left[k \frac{\lambda_o}{B_o}\left(\frac{\partial p_o}{\partial x} - \gamma_o \frac{\partial z}{\partial x}\right)\right] d\Omega \approx (\Upsilon_o)_{i-\frac{1}{2}}\left[(p_o)_{i-1} - (p_o)_i - (\gamma_o)_{i-\frac{1}{2}}(z_{i-1} - z_i)\right] + (\Upsilon_o)_{i+\frac{1}{2}}\left[(p_o)_{i+1} - (p_o)_i - (\gamma_o)_{i+\frac{1}{2}}(z_{i+1} - z_i)\right]$

Transmissibility: $(\Upsilon_o)_{i-\frac{1}{2}} = \left(\frac{k k_{ro}}{B_o \mu_o} \frac{A}{\Delta x}\right)_{i-\frac{1}{2}}$

$$\int_{V_i} \frac{\partial}{\partial t}\left(\phi \frac{S_o}{B_o}\right) \overset{d\Omega}{V} = \frac{V_i}{\Delta t}\left[-(\phi b_o)^{n+1} \Delta_t S_w + (1 - S_w)^n (b_o^{n+1} \phi' + \phi^n b_o') \Delta_t p_o\right]$$

where $b_o' = \frac{d b_o(p_o)}{d p_o} = \frac{b_o^{n+1} - b_o^n}{p_o^{n+1} - p_o^n}$, $\phi' = \frac{d\phi(p_o)}{d p_o} = \frac{\phi^{n+1} - \phi^n}{p_o^{n+1} - p_o^n}$

$\tilde{q}_l$ will be treated in later Chapters.

2° Water conservation equation.

In a cell, $\int_{V_i} \frac{\partial}{\partial x}\left[k \frac{\lambda_w}{B_w}\left(\frac{\partial p_w}{\partial x} - \gamma_w \frac{\partial z}{\partial x}\right)\right] d\Omega$

$\approx (\Upsilon_w)_{i-\frac{1}{2}}\left[(p_o)_{i-1} - (p_o)_i - (\gamma_w)_{i-\frac{1}{2}}(z_{i-1} - z_i)\right] + (\Upsilon_w)_{i+\frac{1}{2}}\left[(p_o)_{i+1} - (p_o)_i - (\gamma_w)_{i+\frac{1}{2}}(z_{i+1} - z_i)\right] - \left[(\Upsilon_w)_{i-\frac{1}{2}} (P_c')_{i-\frac{1}{2}} ((S_w)_{i-1} - (S_w)_i) + (\Upsilon_w)_{i+\frac{1}{2}} (P_c')_{i+\frac{1}{2}} ((S_w)_{i+1} - (S_w)_i)\right.$

$$\int_{V_i} \frac{\partial}{\partial t}\left(\phi \frac{S_w}{B_w}\right) d\Omega = \frac{V_i}{\Delta t}\Big[\left(\left(\phi b_w\right)^{n+1} - S_w^n \phi^n b_w' P_c'\right) \Delta_t S_w + \left(S_w^n b_w^{n+1} \phi' + S_w^n \phi^n b_w'\right) \Delta_t P_o\Big]$$

$$\text{where} \quad b_w' = \frac{d b_w(P_w)}{d P_w} = \frac{b_w^{n+1} - b_w^n}{P_w^{n+1} - P_w^n} \qquad P_c' = \frac{d P_c(S_w)}{d S_w} = \frac{P_c^{n+1} - P_c^n}{S_w^{n+1} - S_w^n}$$

Matrix Form:

$$T\vec{P}^{n+1} + D(\vec{P}^{n+1} - \vec{P}^n) - \vec{G} - \vec{Q} = 0 \left(= \vec{R}\right)$$

- T: matrix of transmissibilities ; dimension $2N \times 2N$; block tridiagonal (2×2)
- D: matrix of accumulation terms ; dimension $2N \times 2N$; block diagonal (2×2)
- $\vec{G}$: vector of gravitational terms
- $\vec{Q}$: vector of well terms
- $\vec{R}$: residual vector
- $\vec{P}$: vector of unknowns ; dimension $2N$

$$\vec{P} = \left[(P_o)_i, (S_w)_i\right]^T \quad (i = 1, \cdots, N) \quad (N \text{ is the number of blocks})$$

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} & & & \\ T_{2,1} & T_{2,2} & T_{2,3} & & \\ & T_{3,2} & T_{3,3} & \ddots & \\ & & \ddots & \ddots & T_{N-1,N} \\ & & & T_{N,N-1} & T_{N,N} \end{pmatrix}$$

$$\begin{cases} T_{i,i-1} = \begin{pmatrix} (\Upsilon_w)_{i-\frac{1}{2}} & -(\Upsilon_w P_c')_{i-\frac{1}{2}} \\ (\Upsilon_o)_{i-\frac{1}{2}} & 0 \end{pmatrix} \\ T_{i,i} = \begin{pmatrix} -(\Upsilon_w)_{i-\frac{1}{2}} - (\Upsilon_w)_{i+\frac{1}{2}} & (\Upsilon_w P_c')_{i-\frac{1}{2}} + (\Upsilon_w P_c')_{i+\frac{1}{2}} \\ -(\Upsilon_o)_{i-\frac{1}{2}} - (\Upsilon_o)_{i+\frac{1}{2}} & 0 \end{pmatrix} \\ T_{i,i+1} = \begin{pmatrix} (\Upsilon_w)_{i+\frac{1}{2}} & -(\Upsilon_w P_c')_{i+\frac{1}{2}} \\ (\Upsilon_o)_{i+\frac{1}{2}} & 0 \end{pmatrix} \end{cases}$$

$$D = \mathrm{diag}\{D_i\} \qquad D_i = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}_i$$

$$\begin{cases} d_{11} = S_w^n\left(b_w^{n+1}\phi' + \phi^n b_w'\right)\frac{V}{\Delta t} \\ d_{12} = \left[(\phi b_w)^{n+1} - S_w^n \phi^n b_w' P_c'\right]\frac{V}{\Delta t} \\ d_{21} = (1 - S_w^n)\left[b_o^{n+1}\phi' + \phi^n b_o'\right]\frac{V}{\Delta t} \\ d_{22} = -(\phi b_o)^{n+1}\frac{V}{\Delta t} \end{cases}$$

Note: The matrix takes one water conservation equation then one oil conservation equation

Fully Implicit Solution: (Newton's method)

$$\cancel{T^{n+1}\vec{P}^{n+1} - D(\vec{P}^{n+1} - \vec{P}^{n}}$$

$$\begin{cases} \left[T^{n+1}\vec{P}^{n+1} - \left(\left(\frac{V}{\Delta t}\Delta_t(\phi b_o S_o)\right)_i\right) - \vec{G}^{n+1} - \vec{Q}\right]^k = \vec{R}^k \\ J_{i,j}^k = \frac{\partial R_i^k}{\partial P_j} \\ J^k \vec{\delta}^{k+1} = -\vec{R}^k \\ \vec{P}^{n+1,k+1} = \vec{P}^{n+1,k} + \vec{\delta}^{k+1} \end{cases}$$

Determination of Transmissibility:

$$(\Upsilon_w)_{i+\frac{1}{2}} = \underbrace{\left(\frac{kA}{\Delta X}\right)_{i+\frac{1}{2}}}_{\text{geometric term}} \underbrace{\left(\frac{k_{rw}}{B_w \mu_w}\right)_{i+\frac{1}{2}}}_{\text{fluid term}}$$

1° Geometric term: $\Delta X_{i+\frac{1}{2}} = X_{i+1} - X_i$

$$k_{i+\frac{1}{2}} = \frac{\Delta X_i + \Delta X_{i+1}}{\frac{\Delta X_i}{k_i} + \frac{\Delta X_{i+1}}{k_{i+1}}}$$

note: $k_{i+\frac{1}{2}}$ is determined in 1D, for steady state, single phase, incompressible flow. This determination is not exact in other cas

2° Fluid term: denote $(H_w)_{i+\frac{1}{2}} = \left(\frac{k_{rw}}{B_w \mu_w}\right)_{i+\frac{1}{2}}$

$$(H_w)_{i+\frac{1}{2}} = \begin{cases} (H_w)_i, & (u_w)_{i+\frac{1}{2}} > 0 \\ (H_w)_{i+1}, & (u_w)_{i+\frac{1}{2}} < 0 \end{cases} \quad \text{upwind}$$

Without gravitational effect,

$$\begin{cases} (P_l)_{i+1} - (P_l)_i > 0 \Rightarrow (u_l)_{i+\frac{1}{2}} < 0 \\ (P_l)_{i+1} - (P_l)_i < 0 \Rightarrow (u_l)_{i+\frac{1}{2}} > 0 \end{cases}$$

With gravitational effect,

denote $\Gamma_l = (P_l)_{i+1} - (P_l)_i - (\gamma_l)_{i+\frac{1}{2}}(z_{i+1} - z_i)$

$$\begin{cases} \Gamma_l < 0, \Rightarrow (u_l)_{i+\frac{1}{2}} > 0 \\ \Gamma_l > 0 \ \Rightarrow \ \ldots \end{cases}$$