van der Pol equ.

1. $x'' + \varepsilon(x^2-1)x' + x = 0$ $(\varepsilon>0)$ (ε is small)

$$\begin{cases} x' = y \\ y' = -\varepsilon(x^2-1)y - x \end{cases}$$

2. $x^* = y^* = 0$

3. $J_{(x^*,y^*)} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}$

$\det|\lambda I - J| = \lambda^2 - \varepsilon\lambda + 1$

$\lambda = \frac{\varepsilon}{2} \pm \sqrt{1-\frac{\varepsilon^2}{4}}\, i \Rightarrow$ unstable focus

(linearize) near curve: $y = \frac{x}{\varepsilon(1-x^2)}$

$\begin{cases} x' = y \\ y' = \varepsilon y - x \end{cases}$ (L) linear

$\begin{cases} x' = y \\ y' = -\varepsilon(x^2-1)y \end{cases}$ (N) non-linear

4. get domain R.

5. check $\underline{CD}$ $\underline{DE}$ $\underline{EA'}$. $\underline{AB}$ $\underline{BC}$

CD: (VdP) $\frac{dy}{dx} = \frac{-\varepsilon(x^2-1)y - x}{y} = 0$

DE: (L) $\frac{dy}{dx} = \varepsilon - \frac{x}{y}$

AB, BC: (N) $\frac{dy}{dx} = -\varepsilon(x^2-1)$

point A is set below E'.

5. stability and Liapunov function.

$x' = f(x)$ limit point:

- ω-limit point: $x = \varphi(t, x_0)$, if x^* is on x, and $\exists \{t_n\}, t_n \to +\infty$, st. $\lim_{n\to\infty} \varphi(t_n, x_0) = x^*$.
- α-limit point: — — $t_n \to -\infty$, — — —.

note: ω-limit point does not necessarily define "stability", and α-limit point — "unstability".

stability of a critical point:

* stable: $\forall \varepsilon > 0, \exists \delta > 0: \forall t > t_0, \forall \|x_0\| < \delta \Rightarrow \|\varphi(t; t_0, x_0)\| < \varepsilon$

* asymptotically stable: $\exists \delta > 0$, st. $\lim_{t\to+\infty} \varphi(t; t_0, x_0) = (0,0)$, $\boxed{\forall \|x_0\| < \delta}$

* exponetially stable: $\exists \alpha > 0, \forall \varepsilon > 0, \exists \delta > 0: \forall \|x_0\| < \delta, \|\varphi(t; t_0, x_0)\| \le \varepsilon e^{-\alpha(t-t_0)}$

Lyapunov function :

$E(X) \geq 0$, $X(t)$, $\frac{dE}{dt}\Big|_{X=X(t)} = \nabla E\Big|_{X=X(t)} \cdot \frac{dX}{dt} \leq 0$

Thm: If $\exists$ Lyapnov func. $E(X)$. to an ODE sys., C^1 to X, nonincreasing along each trajectory $X(t)$.

$E(0) = 0$, $E(X) > 0$, for $X \neq 0$.

$\frac{dE}{dt}\Big|_{X=X(t)} = \nabla E \cdot \dot{X} \leq 0$ in a neighborhood of 0 : $\check{U}(0,\delta)$

Then 0 is stable. If $\frac{dE}{dt} < 0$, $\Rightarrow$ asyp--- stable.

If $E(X) \to +\infty$ for $X \to \infty \Rightarrow$ global stable

6. Bifurcation theory. (bifurcation: controlling parameter changes the structure of the system).

① $x' = \lambda - x^2$

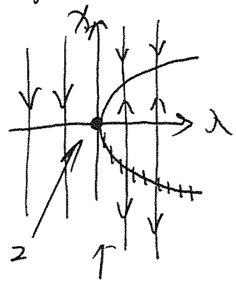
3.

1: bifurcation diagram

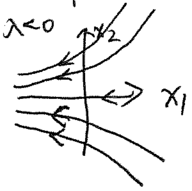
2: bifurcation point

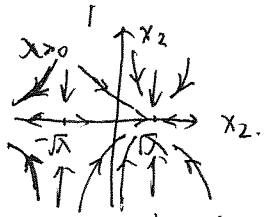
3: controlling parameter

{ tangential bifurcation
{ saddle-node bifurcation.



$$\begin{cases} x_1' = \lambda - x_1^2 \\ x_2' = -x_2 \end{cases}$$



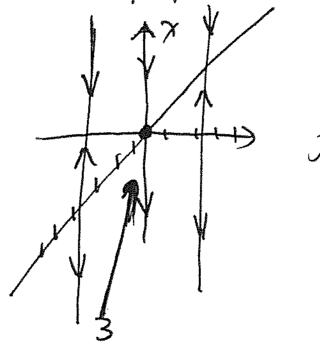


② $x' = \lambda x - x^2$

1: $x^* = 0, \lambda$

2: transcritical bifurcation

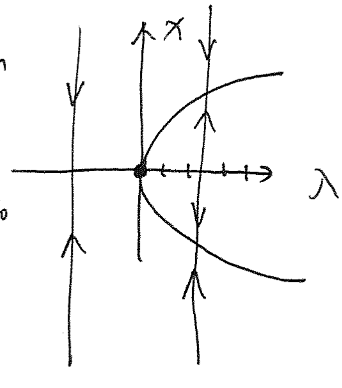
3: exchange of stability



③ $x' = \lambda x - x^3$

1: $x^* = 0, \pm\sqrt{\lambda}$

2: supercritical bifurcation



$$\begin{cases} x' = -x + y(\lambda - x^2 - y^2) \\ y' = y + x(\lambda - x^2 - y^2) \end{cases}$$

Hopf bifurcation.

(bifurcation with respect to closed orbits).

$\Downarrow$

$$\begin{cases} r' = r(\lambda - r^2) \\ \theta' = 1 \end{cases}$$

7、 chaos.

$$\begin{cases} X' = -\sigma X + \sigma Y \\ Y' = rX - Y \boxed{-XZ} \\ Z' = -bZ \boxed{+XY} \end{cases}$$

Symmetry: $\{X, Y, Z\} \to \{-X, -Y, Z\}$

strange attractor

$$V(X, Y, Z) = rX^2 + \sigma Y^2 + \sigma (Z - 2r)^2$$

$$\frac{dV}{dt} = 2rX \cdot X' + 2\sigma Y \cdot Y' + 2\sigma (Z - 2r) \cdot Z'$$

$$= -2\sigma (rX^2 + Y^2 + bZ^2 - 2rbZ)$$

(< 0, under some condition)

Critical point: $X(r - 1 - \frac{X^2}{b}) = 0$

① $X = 0$, $Y = 0$, $Z = 0$

② $\begin{cases} X = \pm\sqrt{b(r-1)} \quad (r \geq 1) \\ Y = \pm\sqrt{b(r-1)} \\ Z = r - 1 \end{cases}$

$(0,0,0)$: $J = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$

$r<1$, stable; $r>1$, saddle

$(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1)$: $J = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{b(r-1)} \\ \sqrt{b(r-1)} & \sqrt{b(r-1)} & -b \end{pmatrix}$

Logistic map

$$x_{n+1} = \lambda x_n (1-x_n)$$

$\lambda \le 1$, $1<\lambda<3$ ~~$\lambda>1$~~, $\lambda=4$.

2-periodic solution

$\lambda=4$, $\begin{cases} x_{n+1} = 4x_n(1-x_n) & (x_n \in [0,1]) \\ x = \sin^2 \frac{\pi y}{2} & (\text{mapping}) \end{cases}$

$\Rightarrow \sin^2 \frac{\pi y_{n+1}}{2} = \sin^2 \pi y_n \quad (y_n \in [0,1])$

$\Rightarrow$ ① $y_{n+1} = 2y_n \pmod 1 \quad (y_n \in [0,1])$

or ② $y_{n+1} = \begin{cases} 2y_n, & 0 \le y_n \le \frac{1}{2} \\ 2(1-y_n), & \frac{1}{2} \le y_n \le 1 \end{cases}$

①
$\bar{y}_0 = 0.1011001110101\cdots$
$\tilde{y}_0 = 0.1011001110100\cdots$

②

when $|f'(x)| = |\lambda(1-2x)| < 1$, say $\lambda \in (1,3)$ and

$|f'(1-\frac{1}{\lambda})| = |2-\lambda| < 1$

then near $x^* = 1-\frac{1}{\lambda}$, $\left|\frac{x_{n+2}-x_{n+1}}{x_{n+1}-x_n}\right| = \left|\frac{f(x_{n+1})-f(x_n)}{x_{n+1}-x_n}\right| \approx |f'(x_n)| < 1$

and $(x_{n+1}-x_n)\cdot(x_{n+2}-x_{n+1}) < 0$.

$\Rightarrow$ $\{x_n\}$ converges to $x^* = 1-\frac{1}{\lambda}$.