Ordinary Least Square (OLS) estimator

No.

Date

最小二乘法

参数(未知数)个数 < 方程(点)个数.

如,已知 $(n+1)$ 个点 (x_i, y_i) $(i=0,\cdots,n)$

多项式 $y = a_0 + a_1 x + \cdots + a_m x^m$ ($m+1$ 个参数),且 $m<n$.

建立方程组:

$$\left.\begin{array}{l} a_0 + a_1 x_0^1 + \cdots + a_m x_0^m - y_0 = \delta_0 \\ \cdots\cdots \\ a_0 + a_1 x_n^1 + \cdots + a_m x_n^m - y_n = \delta_n \end{array}\right\} n+1\text{个方程}$$

$(\delta_0, \delta_1, \cdots, \delta_n$ 表示误差)

我们的目标是找到 $(a_0, a_1, \cdots, a_m)$,使得 $\sum_{i=0}^{n} \delta_i^2$ 最小.(以致 $|\delta_i|$ 很小).

令 $G = (x_i^j)_{(n+1)\times(m+1)}$ $(i=0,\cdots,n;\ j=0,\cdots,m)$

$a = (a_0, a_1, \cdots, a_m)^T_{m+1}$

$y = (y_0, y_1, \cdots, y_n)^T_{n+1}$

$\delta = (\delta_0, \delta_1, \cdots, \delta_n)^T_{n+1}$

则 $G \cdot a - y = \delta$,求 a 使 $\|\delta\|$ 最小.

其中 $\|\delta\|^2 = [Ga-y]^T \cdot [Ga-y]$,视为 $f(a)$,对 a 求导.(*)

$G^T \cdot G \cdot a - G^T y = 0$.

可证 $|G^T \cdot G| > 0$.

$\therefore\ a = (G^T G)^{-1} \cdot G^T y$.　　投影矩阵 $G(G^T G)^{-1} G^T_{(n+1)\times(n+1)}$

(*): 对求导的补充. 设 $f(a) = [Ga-y]^T[Ga-y] = \sum_{i=0}^{n}\left(\sum_{j=0}^{m} x_i^j a_j - y_i\right)^2$

则 $\frac{\partial f}{\partial a_k} = \sum_{i=0}^{n} 2\cdot\left(\sum_{j=0}^{m} x_i^j a_j - y_i\right)\cdot x_i^k$,令 $\frac{\partial f}{\partial a_k} = 0$

则 $\sum_{i=0}^{n}\left(\sum_{j=0}^{m} x_i^j a_j\right) x_i^k - \sum_{i=0}^{n} y_i x_i^k = 0$ $(k=0,\cdots,m)$

把方程组按列排列有 $G^T G \cdot a - G^T \cdot y = 0$