Open Mapping Theorem

We first prove a preliminary lemma that does all the hard work.

Lemma Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear transformation mapping X *onto* Y. Then there exists a $d > 0$ such that the image of the open unit ball $B_1(0) \subset X$ contains the open ball $B_d(0) \subset Y$, i.e.

$$TB_1(0) \supset B_d(0). \tag{1}$$

Proof Since $\cup_1^\infty B_n(0) = X$ and $T(X) = Y$,

$$\bigcup_1^\infty TB_n(0) = Y,$$

and therefore by Baire's Theorem, for some n_0, TB_{n_0} is dense in some open set $\mathcal{V} \subset Y$. Thus there is some $y_0 \in \mathcal{V}$ and $x_0 \in B_{n_0}$, with $Tx_0 = y_0$, such that

$$T(B_{n_0} - x_0) \quad \text{is dense in} \quad \mathcal{V} - y_0,$$

an open set containing 0.

The set $B_{n_0} - x_0$ is contained in the ball, $B_{n_0+|x_0|}$. By homogeneity of T, i.e. $T(\alpha x) = \alpha T(x)$,

$$TB_1(0) \quad \text{is dense in} \quad B_r(0) \subset Y,$$

for some $r > 0$. For the same reason, for any $c > 0$,

$$TB_c(0) \quad \text{is dense in} \quad B_{cr}(0) \subset Y. \tag{2}$$

Next we show $TB_2(0) \supset B_r(0)$, i.e. given $u \in B_r(0) \subset Y$, $\exists x \in X$ such that,

$$Tx = u.$$

The point x is constructed as the sum of a series,

$$x = \sum_1^\infty x_j.$$

It follows from (2) with $c = 1$, there is an x_1 satisfying

$$\|u - Tx_1\| < \frac{r}{2}, \qquad \|x_1\| < 1.$$

Next, from (2) with $c = \frac{r}{2}$, there is an x_2 satisfying

$$\|u - Tx_1 - Tx_2\| < \frac{r}{4}, \qquad \|x_2\| < \frac{1}{2}.$$

In general, with $c = \frac{1}{2^{m-1}}$, there is an x_m satisfying

$$\|u - \sum_1^m Tx_j\| < \frac{r}{2^m}, \qquad \|x_m\| < \frac{1}{2^{m-1}}. \tag{3}$$

It follows that the series for x converges since

$$\|x\| \le \sum \|x_j\| < \sum_1^\infty \frac{1}{2^{j-1}} = 2.$$

Finally, from (3),

$$u = \sum_1^\infty Tx_j = T\sum_1^\infty x_j = Tx,$$

and the Lemma is proved. ■

Corollary 1 - The Open Mapping Theorem Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear transformation mapping X *onto* Y. Then T is an open mapping, i.e. $T\mathcal{U}$ is open in Y for every open $\mathcal{U} \subset X$.

Proof The Lemma implies $TB_1(0) \supset B_d(0)$ for some $d > 0$. Let $y \in TB_1(0)$ and $x \in B_1(0)$ such that $Tx = y$. Tale $\alpha > 0$ so small that $\alpha x \in T^{-1}B_d(0)$. Then $\alpha y = \alpha Tx \in B_d(0)$. Then choose $\delta > 0$ so that

$$TB_\delta(\alpha y) \subset B_d(0) \quad \text{and} \quad B_\delta(\alpha x) + (x - \alpha x) \subset B_1(0).$$

Then

$$B_\delta(\alpha y) + (y - \alpha y) = B_\delta(y) \subset TB_1(0),$$

and the theorem is proved. ■

Corollary 2 Let X and Y be Banach spaces and $T : X \to Y$ a bounded linear $1-1$ transformation mapping X *onto* Y. Then the algebraic inverse $T^{-1} : Y \to X$ is bounded.

Proof From (1) in the lemma, for each $y \in Y$ such that $\|y\| = d/2$ there exists $x \in B_1(0)$ such that $Tx = y$. Note

$$\|x\| \leq 1 = 2\|y\|/2. \tag{4}$$

Also from the $1-1$ property of T, $x = T^{-1}y$ and (4) implies $\|T^{-1}y\| \leq \frac{2}{d}\|y\|$, i.e. T is bounded. ■

Corollary 3 - Closed Graph Theorem Let X and Y be Banach spaces and T a closed linear transformation having domain $\mathcal{D}(T) = X$ and range $\mathcal{R}(T) \subset Y$.
Then T is bounded.

Proof Let $G_T \subset X \times Y$ be the graph of T and define the *graph norm* of $g = (x, Tx) \in G_T$ to be,

$$\|g\|_{G_T} = \|x\|_X + \|Tx\|_Y.$$

Next note that G_T is closed in $X \times Y$ and thus is complete. Define the projection $P : G_T \to X$,

$$P(x, Tx) = x.$$

P is clearly bounded, linear and maps G_T in a $1-1$ manner onto X. Corollary 2 then implies $P^{-1} : X \to G_T$ is bounded. But

$$P^{-1}(x) = (x, Tx),$$

and the Corollary is proved. ■

If the "closed" assumption in Corollary 3 is omitted one can find a counterexample:

Lemma

Given normed linear spaces X and Y, $Y \neq \{0\}$ and dim X infinite, there exists a linear transformation $L : X \to Y$ with $\mathcal{D}(L) = X$ but L is not bounded and thus G_L cannot be closed.