Defn : A, B Hermitian,

$A \geq B \iff A-B$ posi semi def

$A > B \iff A-B$ pos def

1° Reflexive $A \geq B$

2° Equality $A \geq B$ & $B \geq A \Rightarrow A = B$

3° Transivity $A \geq B,\ B \geq C \Rightarrow A \geq C$

Note: 1° This is a partial ordering. (No all Hermitian are comparable.)

2° $A \geq B,\ A \neq B \not\Rightarrow A > B$

Lemma: A, B Hermitian, then

(1) $A \geq B \Rightarrow T^*AT \geq T^*BT$, $T \in M_{n,m}$

(2) $A > B \Rightarrow T^*AT > T^*BT$, T full rank (row ?)

Thm : A, B Hermitian, If $\exists$ $\mathbb{R}$-linear combination of A & B that is pos. def

$\Rightarrow \exists$ non-sing. C s.t. both C^*AC & C^*BC diagonal.

Thm : A, B Hermitian, A posi def, B pos semi def,

$\Rightarrow \{A \geq B \iff \rho(BA^{-1}) \leq 1\}$ and

$\{A > B \iff \rho(BA^{-1}) < 1\}$

Coroll : A, B pos semi-def :

(a) $A \geq B \iff$ (pos def) $B^{-1} \geq A^{-1}$

(b) $A \geq B \Rightarrow \det(A) \geq \det(B)$, $\mathrm{tr}(A) \geq \mathrm{tr}(B)$

(c) $A \geq B \Rightarrow \lambda_k(A) \geq \lambda_k(B)$ $(\lambda_1 \geq \cdots \geq \lambda_n)$

Thm: $H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$, A, C Hermitian, then

H pos def $\Leftrightarrow$ A pos def & $C > B^* A^{-1} B$

$$H^{-1} = \begin{bmatrix} (A - BC^{-1}B^*)^{-1} & A^{-1}B(B^*A^{-1}B - C)^{-1} \\ (B^*A^{-1}B - C)^{-1} B^* A^{-1} & (C - B^*A^{-1}B)^{-1} \end{bmatrix}$$

Thm: $A \in M_n$, $C \in M_m$, pos def, $B \in M_{n,m}$, TFAE:

(1) $(x^*Ax)(y^*Cy) \geq |x^*By|^2$, $\forall x \in \mathbb{C}^n, y \in \mathbb{C}^m$.

(2) $(x^*Ax) + (y^*Cy) \geq 2|x^*By|$, $\forall x \in \mathbb{C}^n, y \in \mathbb{C}^m$

(3) $\rho(B^*A^{-1}BC^{-1}) \leq 1$

(4) $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$.

Thm: P pos def, $S \subseteq [n]$, then

$(P[S])^{-1} \leq P^{-1}[S]$

Thm: A, B pos def., then

$A^{-1} \circ B^{-1} \geq (A \circ B)^{-1}$

When $B = A^{-1}$,

$$A^{-1} \circ A \geq \underset{(A^{-1}\circ A)}{I} \geq (A^{-1} \circ A)^{-1}.$$