≪ Intermediate Classical Mechanics ≫

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Driven Oscillations

1. Sinusoidal Driving Forces

1.1. equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos wt.$$

using another notation,

$$\ddot{x} + 2\gamma\dot{x} + w_0^2 x = A \cos wt$$

where $2\gamma = \frac{c}{m}$, $w_0^2 = \frac{k}{m}$, $A = \frac{F_0}{m}$.

the solution of the abovt equation consists of two parts, a complementary function $x_c(t)$ and a partlcular function $x_p(t)$.

1.2. now we solve $x_p(t)$ using complex numbers.

let $x_i(t) = Re(Ce^{iwt})$ (where C is complex),

and write the driving term as $Re(F_0 e^{iwt})$

then

$$\cancel{Re(-w^2 C e^{iwt}) + 2\gamma Re(iwCe^{iwt}) + w_0^2 Re(Ce^{iwt}) = Re(Ae^{iwt})}$$

$$(-w^2 + 2i\gamma w + w_0^2)\, Ce^{iwt} = \frac{F_0}{m} e^{iwt}$$

implying

$$C = \frac{F_0/m}{-w^2 + 2i\gamma w + w_0^2}$$

$$= \frac{F_0/m\,(w_0^2 - w^2 - 2i\gamma w)}{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}$$

$$= Ae^{-i\phi}$$

where amplitude: $A = \frac{F_0/m}{\sqrt{(w_0^2 - w^2)^2 + 4\gamma^2 w^2}}$

phase: $\phi = \tan^{-1} \frac{2\gamma w}{w_0^2 - w^2}$ $(\phi \in [0, \pi])$

then the particular solution is

$$x_i(t) = Re(Ce^{iwt}) = Re(Ae^{i(wt - \phi)}) = A\cos(wt - \phi)$$

x_i is called Steady State Solution

1.3. Amplitude of Resonance

resonance frequency is

$$w_R = \sqrt{w_0^2 - 2\gamma^2}$$

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the resonance frequency is lower than the damping ~~coefficients~~ frequency

the maximum amplitude.

$$A = \frac{F_0/m}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$

in case of small ~~damping~~ damping

$$A \simeq \frac{F_0}{2m\gamma\omega_0} = \frac{F_0}{C\omega_0} \quad (\text{if } \gamma \to 0,\ A \to +\infty)$$

1.4 quality factor

it's customary to describe the degree of damping in an oscillating system in terms of the "quality factor".

$$Q \equiv \frac{\omega_R}{2\gamma}$$

when γ increases, Q decreases.

when γ is small, or Q is very large, the shape of resonance curve approaches that for an undamped oscillator;

when γ is large, or Q is very small, the resonance can be completely destroyed.

if γ is large, or Q is very small, the resonance frequency ω_R lowers, and the maximum amplitude is small, and δ, which shows the delay, gets larger.

1.5 energy

the total mechanical energy is time dependent (steady state solution)

$$\begin{aligned}
E &= K + V \\
&= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 \\
&= \tfrac{1}{2}mA^2\omega^2\sin^2(\omega t-\phi) + \tfrac{1}{2}kA^2\cos^2(\omega t-\phi) \\
&= \tfrac{1}{2}mA^2\left[\omega^2\sin^2(\omega t-\phi) + \omega_0^2\cos^2(\omega t-\phi)\right] \\
&= \tfrac{1}{2}mA^2\omega_0^2 + \tfrac{1}{2}mA^2(\omega^2-\omega_0^2)\sin^2(\omega t-\phi)
\end{aligned}$$

when ω is close to ω_0, the additionary component is small, then the total energy stays at a level and oscillates with a small amplitude, and the period is $\frac{\pi}{\omega}$, half of that of the diven oscillation main frequency ?

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1-6* kinetic energy resonance.

$$K = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t - \phi)$$

the average kinetic energy over a period is

$$\bar{K} = \frac{1}{4} m \omega^2 A^2$$

let $\frac{d\bar{K}}{d\omega} = 0$, then

$$\omega_E = \omega_0$$

that is to say, the kinetic energy resonance occurs at the natural frequency of the system of undamped oscillations.

1-7 fourier series.

a periodic function can always be written as a series of many sinusoidal functions.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

where $\omega = \frac{2\pi}{T}$, and T is the period.

where the coefficients a_n and b_n are determined by the follows.

$$a_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \cos(n\omega t)\, dt \quad , \text{ for } n = 0, 1, 2, \cdots$$

$$b_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} f(t) \sin(n\omega t)\, dt \quad , \text{ for } n = 1, 2, 3, \cdots$$