Coupled Oscillations and Normal Modes.

1° $\mathcal{L} = T - V$

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \quad (i = 1, 2, \cdots, N)$ n particles.

2° guess normal mode exists, substitute x_i with $a_i \cos(\omega t - \delta)$ then get a set of linear equations of a_i.

3° let the coefficient determinant be zero to get the non-trivial solution according to different ω.

4° the general solution will be the linear combination of the n normal modes.

General Theorey of Vibrating Systems.

1° Now we discuss the general case that a system with n degrees of freedom that is oscillating about a point of equilibrium.

We specify the configuration by n generalized coordinates $q_1, q_2, \cdots, q_n$.

$\vec{q} = (q_1, q_2, \cdots, q_n)$.

we assume the system is conservative, and it has a potential energy.

$U(q_1, q_2, \cdots, q_n) = U(\vec{q})$

we assume the constraint here is constant, hence the displacement of the α-th particle is. $\vec{r}_\alpha = \vec{r}_\alpha(q_1, q_2, \cdots, q_n)$

Then T can be written in the form:

$$T = T(\vec{q}, \dot{\vec{q}}) = \frac{1}{2}\sum_{j,k} A_{jk}(\vec{q})\, \dot{q}_j \dot{q}_k$$

2° The Taylor Expansion of $U(\vec{q})$ is (appoximation)

$$U(\vec{q}) = U(0) + \sum_j \frac{\partial U}{\partial q_j} q_j + \frac{1}{2}\sum_{j,k} \frac{\partial^2 U}{\partial q_j \partial q_k} q_j q_k + \cdots$$

let $U(0) = 0$, and for $\vec{q} = 0$ is the equilibrium point, hence $\frac{\partial U}{\partial q_j}(0) = 0$ for any j.

We now take the second order terms only, and get

$$U(\vec{q}) = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_k.$$

let $\vec{q} = 0$ and the T is just the second order terms:

$$T(\dot{\vec{q}}) = \frac{1}{2}\sum_{jk} M_{jk} \dot{q}_j \dot{q}_k, \text{ where } M_{jk} = A_{jk}(0).$$

Hence the Lagrangian,

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}) = T(\dot{\vec{q}}) - U(\vec{q}).$$

3° (Equation of Motion in Matrix form).

$$\frac{\partial U}{\partial q_i} = \sum_j k_{ij} q_j \quad (i = 1, \cdots, n).$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \sum_j M_{ij} \ddot{q}_j \quad (i = 1, \cdots, n).$$

Hence, the Lagrange's Equations are

$$\sum_j M_{ij} \ddot{q}_j = -\sum_j k_{ij} q_j \quad [i = 1, 2, \cdots, n]$$

in Matrix form is, $\mathbf{M}\ddot{\vec{q}} = -\mathbf{K}\vec{q}$, where $\mathbf{M}$ and $\mathbf{K}$ are "mass" and "spring-constrant" matices.

4° (Method of Normal mode) try solutions in the form of $\vec{q} = \vec{a}\cos(\omega t - \delta)$, we get the eigenvalue equation,

$$(\mathbf{K} - \omega^2 \mathbf{M})\vec{a} = 0$$

then let ω satisfies the characteristic (or secular) equation,

$$\det(\mathbf{K} - \omega^2 \mathbf{M}) = 0$$

If we get n different non-negative ω, and $\vec{a}$ respectively, the general solution is just the linear combination of the normal mode solutions.