Nov 7.

Migration: $p_i^{(m)}$: population in city i, at step m.

a_{ij}: ratio of population moving from city j to city i.

$$p_i^{(m+1)} = \sum_j a_{ij} p_j^{(m)}$$

$$p^{(m+1)} = A p^{(m)}$$

$$p^{(m+1)} = A^{m+1} p^{(0)}$$

$$\sum_i a_{ij} = 1 \quad , \forall j=1,2,\cdots,n.$$

$$0 \le a_{ij} \le 1 \quad , \forall i,j.$$

Goal: Find $\underbrace{\text{long-term}}_{\text{(asymptotic)}}$ distribution of population. (limit of p).

Def: $A \in M_{m,n}$, real, is nonegative (positive), iff all entries are nonegative (positive).

Denote by $A \geq 0$ $(A > 0)$.

Def: $A, B \in M_{m,n}$.

$$A \geq B, \text{ iff } A - B \geq 0$$
$$A > B, \text{ iff } A - B > 0$$

Fun facts: (1) $|A| \geq 0$ and $\begin{cases} \exists x > 0, \text{s.t.} \\ |A|x = 0 \end{cases} \Leftrightarrow A = 0$

(2) $|aA| = |a||A|$, $\forall a \in \mathbb{C}$

(3) $|A+B| \leq |A| + |B|$

(4) $A \geq 0$ & $A \neq 0 \Rightarrow A > 0$. only when $m=n=1$

(5) $A \geq 0, B \geq 0$, $a, b \geq 0 \Rightarrow aA + bB \geq 0$

(6) $A \geq B$, $B \geq C \Rightarrow A \geq C$

entry-wise absolute value $|A| = [|a_{ij}|]$

Prop: (a) $|AX| \le |A|\,|X|$

(b) $A \ge 0$ w/ positive row.

$|AX| = A|X| \Rightarrow \exists \theta \in [0, 2\pi)$, s.t. $e^{-i\theta} X = |X|$

(c) If X positive, ~~$AX = |A|X$~~

$AX = |A|X \Rightarrow A \ge 0$.

$\sum_j (|a_{ij}| - Re\{a_{ij}\}) x_j = 0$ $(x_j > 0)$

Fun facts: (1) $|AB| \le |A|\,|B|$, $|A^m| \le |A|^m$

(2) $0 \le A \le B$, & $0 \le C \le D$, then

$0 \le AC \le BD$

$0 \le A \le B \Rightarrow 0 \le A^m \le B^m$

(3) $A > 0$, $X \ge 0$, $X \ne 0 \Rightarrow AX > 0$

(4) $A \ge 0$, $X > 0$, $AX = 0 \Rightarrow A = 0$

(5) $|A| \le |B| \Rightarrow \||A|\| \le \||B|\|$, $\forall$ abs vector norm.

Thm: $A, B \in M_n$, B nonneg,

If $|A| \le B$, then $\rho(A) \le \rho(|A|) \le \rho(B)$

proof: $\lim_{m\to\infty} \|A^m\|^{\frac{1}{m}} = \rho(A)$.

Corol: ~~A, B nonneg~~

If $0 \le A \le B$, then $\rho(A) \le \rho(B)$.

Corol: $A \ge 0$, then

(a) If $\hat{A}$ principal minor, then $\rho(\hat{A}) \le \rho(A)$

(b) $\max_i a_{ii} \le \rho(A)$

(c) $\rho(A) > 0$, if $\forall$ main diag. entry pos.

Lemma: $A \in M_n$, nonneg, then

$\rho(A) \le \max_i \sum_{j=1}^n a_{ij}$ and

$\rho(A) \le \max_j \sum_{i=1}^n a_{ij}$

If all row sums equal, $\rho(A) = $ row sum.

If all column sums equal, $\rho(A) = $ column sum

Thm: $A \in M_n$, nonneg.

$$\min_i \sum_j a_{ij} \le \rho(A) \le \max_i \sum_j a_{ij}$$

$$\min_j \sum_i a_{ij} \le \rho(A) \le \max_j \sum_i a_{ij}$$

Corol: $A \in M_n$, If $A \ge 0$ and either $\sum_j a_{ij} > 0, \forall i$

or $\sum_i a_{ij} > 0, \forall j$

then $\rho(A) > 0$.

In particular, if $n \ge 2$, and A irreducible & nonneg, then $\rho(A) > 0$.

Def: $A \in M_n$ reducible, iff

$\exists$ permutation matrix P, s.t.

$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$, $B \in M_r$, $1 \le r < n$.

Thm: $A \ge 0$, $x > 0$, then

$$\min_i \frac{1}{x_i} \sum_j a_{ij} x_j \le \rho(A) \le \max_i \frac{1}{x_i} \sum_j a_{ij} x_j.$$

$$\min_j x_j \sum_i \frac{a_{ij}}{x_i} \le \rho(A) \le \max_j x_j \sum_i \frac{a_{ij}}{x_i}$$

Nov. 12

Corol 1: $A \ge 0$, $x > 0$, If $\alpha, \beta \ge 0$ s.t. $\alpha x \le Ax \le \beta x$, then

$\alpha \le \rho(A) \le \beta$

If $\alpha x < Ax$, then $\alpha < \rho(A)$;

If $Ax < \beta x$, then $\rho(A) < \beta$.

Corol 2: $A \ge 0$, $x > 0$ eigenvector, then ~~$(\rho(A), x)$ is an eigenvector pair~~

$Ax = \rho(A) x$.

Corol 3': If $A \geq 0$ has a posi eigenvector, then

$$\rho(A) = \max_{x>0} \min_{i} \frac{1}{x_i} \sum_j a_{ij} x_j$$

$$= \min_{x>0} \max_{i} \frac{1}{x_i} \sum_j a_{ij} x_j$$

Theorem: (Perron)

$A \in M_n$, $A > 0$,

(a) $\rho(A) > 0$

(b) $\rho(A)$ is an algebraically simple eigenvalue of A

(c) $\exists !\ x \in \mathbb{R}^n$, s.t. $Ax = \rho(A) x$, and $\sum_i x_i = 1$; $x > 0$

(d) $\exists !\ y \in \mathbb{R}^n$, s.t. $y^T A = \rho(A) y^T$, and $\sum_i x_i y_i = 1$.
Also, $y > 0$

(e) $|\lambda| < \rho(A)$, $\forall$ eigenvalues λ of A, $\lambda \neq \rho(A)$

(f) $(\rho(A)^{-1} A)^m \to x y^T$ as $m \to \infty$.

Theorem: If $A \in M_n$, $A > 0$, then
$\exists\ x, y$, s.t. $Ax = \rho(A) x$, and $y^T A = \rho(A) y^T$

Lemma: $A \in M_n$, $A > 0$, If (λ, x) eigen pair, and $|\lambda| = \rho(A)$,
then $\exists\ \theta \in \mathbb{R}$, st, $e^{-i\theta} x = |x| > 0$.

Thm: $A \in M_n$, $A > 0$, Geometric multi of $\rho(A)$ is 1.

Lemma: $A \in M_n$, $\lambda \in \mathbb{C}$, $x, y \in \mathbb{C}^n \setminus 0$,
Suppose λ geometric mult 1, and $Ax = \lambda x$, $y^* A = \lambda y^*$,
Then $\exists\ \gamma \in \mathbb{C} \setminus 0$, s.t. $\operatorname{adj}(\lambda I - A) = \gamma x y^*$

Thm: $A \in M_n$, $\lambda \in \mathbb{C}$, $x, y \in \mathbb{C}^n \setminus 0$.

Suppose $Ax = \lambda x$ & $y^* A = \lambda y^*$.

If λ has geom mult 1, then

λ has alg. mult 1. $\Leftrightarrow$ $y^* x \neq 0$.

Thm: $A \in M_n$, $\lim_{m \to \infty} A^m = 0$, iff $\rho(A) < 1$.

Such a matrix is called convergent.

Lemma: $A \in M_n$, $x, y \in \mathbb{C}^n \setminus 0$, $\lambda \in \mathbb{C}$, s.t. $Ax = \lambda x$ & $y^* A = \lambda y^*$, $y^* x = 1$, then

$\exists$ nonsingular $S \in M_n$ of form $[x, S_1]$ s.t. $(S^{-1})^* = (S^*)^{-1} = [y \ z_1]$.

and $A = S \begin{bmatrix} \lambda & 0 \\ 0 & B \end{bmatrix} S^{-1}$ for some $B \in M_{n-1}$.

Thm: $A \in M_n$, $A > 0$, If x, y right & left Perron vectors of A,

then $\lim_{m \to \infty} (\rho^{-1}(A) A)^m = xy^T$, positive rank 1 matrix.

Nov. 19

(If $\rho(A)$ an algebraically simple eigenvalue of A, say irreducible nonnegative matrices)

Terminology: $\rho(A)$ — Perron root of A

x — (right) Perron vector of A

y — left Perron vector of A.

Upper bound for convergence:

$$\|(\rho(A)^{-1} A)^m - xy^T\|_\infty = \|S \begin{bmatrix} 1 & 0 \\ 0 & (\rho(A)^{-1} B)^m \end{bmatrix} S^{-1}\|_\infty$$

$$\leq C r^m$$

where $C = C(r, A) > 0$ is a constant,

$r \in \left(\frac{|\lambda_2|}{\rho(A)}, 1 \right)$

$$\frac{|\lambda_2|}{\rho(A)} \le \frac{1-k^2}{1+k^2}, \quad k = \frac{\min\{a_{ij}\}}{\max\{a_{ij}\}}.$$

Thm: (Ky Fan) $A \in M_n$, $B \in M_n$, $B \ge 0$, $b_{ij} \ge |a_{ij}|$ $\forall i \ne j$, then every eigenvalue of A lies in the union of n disc:

$$\bigcup_{i=1}^{n} \{ z \in \mathbb{C} \mid |z - a_{ii}| \le \rho(B) - b_{ii} \}.$$

In particular, if $|a_{ii}| > \rho(B) - b_{ii}$ $\forall i$, then A nonsingular

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For nonnegative matrices:

Thm: $A \ge 0$, then

$\rho(A)$ is an eigenvalue of A, and

$\exists$ nonneg nonzero vector X, s.t. $AX = \rho(A)X$.

Thm: A nonneg, X nonneg nonzero, If $\alpha \in \mathbb{R}$, s.t. $\alpha X \le AX$, then

$\alpha \le \rho(A)$

Corol: If $A \in M_n$, nonneg, then

$$\rho(A) = \max_{\substack{X \ge 0 \\ X \ne 0}} \min_{\substack{1 \le i \le n \\ x_i \ne 0}} \frac{1}{x_i} \sum_{j=1}^{n} a_{ij} x_j$$

Thm: $A \in M_n$, nonneg, If $\exists$ X positive, λ nonneg, st. either $AX = \lambda X$ or $X^T A = \lambda X^T$, then

$\lambda = \rho(A)$

Thm: A nonneg, if A has a positive eigenvector, then

(a) if $x \in \mathbb{R}^n \setminus 0$, and $Ax \geq \rho(A)x$, then x is eigenvector assoc. to $\rho(A)$

(b) if $A \neq 0$, then $\rho(A) > 0$, and every eigenvalue λ, $|\lambda| = \rho(A)$, is semisimple (all Jordan blocks are 1×1).

Thm: $A \in M_n$ power bounded iff every eigenvalue of A has modulus at most 1 and every eigenvalue of modulus 1 is semisimple.

Nov. 26

Def: A matrix $A \in M_n$ is <u>reducible</u> if $\exists$ permutation matrix $P \in M_n$, s.t.

$$P^T A P = \begin{array}{c} \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \\ {\scriptstyle n-r \quad r} \end{array} \qquad (1 \leq r \leq n-1)$$

A is <u>irreducible</u> if it's not reducible.

Idea of the concept of reducible matrix:

If A reducible, let $\hat{A} = P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$,

then $AX = b \iff \hat{A}(P^T X) = (P^T b)$ which is essentially two low order problems

Lemma 1: $A \in M_n$ nonnegative, then

$$A \text{ irreducible} \Leftrightarrow (I+A)^{n-1} > 0$$

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proof: (proof inspired from Markov chains)

Equivalent to proof:

A reducible $\Leftrightarrow$ $(I+A)^{n-1}$ has a zero entry (at least).

"$\Rightarrow$" $\hat{A}^k$ $(k=1,2,\cdots,n-1)$ has a $(n-r)\times r$ zero matrix in the lower left corner.

$$P^T(I+A)^{n-1}P = (I+\hat{A})^{n-1} = I+(n-1)\hat{A}+\cdots+\hat{A}^{n-1}$$

has a $(n-r)\times r$ zero matrix in the lower left corner.

"$\Leftarrow$" A has a zero entry (at least) on the off-diagonal part.

Lemma 2 $A \in M_n$, eigenvalues $\lambda_1, \cdots, \lambda_n$, then

$I+A$ eigenvalues $\lambda_1+1, \cdots, \lambda_n+1$,

and $\rho(I+A) \leq \rho(A)+1$.

If A nonneg, then $\rho(I+A) = \rho(A)+1$.

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proof: If A nonneg, then $\rho(A)$ is an eigenvalue.

Lemma 3: If $A \in M_n$, nonneg and $A^m > 0$, for some $m = 1$, then $\rho(A)$ is the only eigenvalue of maximum modulus.
i.e. it's pos (eigenvalue) & algebraically simple.

<u>Thm</u> (Perron – Frobenius)

$A \in M_n$ irreducible, nonneg & $n \geq 2$, then

(a) $\rho(A) > 0$

(b) $\rho(A)$ is an algebraically simple eigenvalue

(c) $\exists!$ $x \in \mathbb{R}^n$ s.t. $Ax = \rho(A)x$ & $\sum_i x_i = 1$; $x > 0$

(d) $\exists!$ $y \in \mathbb{R}^n$ s.t. $y^T A = \rho(A) y^T$ & $\sum_i x_i y_i = 1$; $y > 0$.

<u>Nov. 28</u>

<u>Thm</u>: $A, B \in M_n$, A nonneg, irreducible, $A \geq |B|$.
Let $\lambda = e^{i\varphi}\rho(B)$ a max modulus eigenvalue of B.
If $\rho(A) = \rho(B)$, then $\exists$ diagnal unitary D s.t.
$$B = e^{i\varphi} DAD^{-1}.$$

o <u>Corol</u>: $A \in M_n$, nonneg, irreducible, with exactly k distinctive eigenvalues of max modulus

(a) A is similar to $e^{2\pi i \, p/k} A$

(b) max modulus eigenvalues of A are $e^{2\pi i \, p/k} \rho(A)$ $(p = 0, 1, \cdots, k-1)$ and each has algebraic mult. 1.

Corol: $A \in M_n$ nonneg, irreducible,

If A has $k > 1$ eigenvalues of max modulus,

then every main diagonal entry of A^m is zero for any positive integer m not divisible by k.

Def: A nonneg matrix A is called primitive if it's irreducible and has only one max modulus eigenvalue.

Thm: $A \in M_n$ nonneg., primitive,

then

$$\lim_{m\to\infty} \left(\rho(A)^{-1}A\right)^m = x y^T$$

where x, y are right, left perron vectors.

Thm: $A \in M_n$ nonneg.

$$A \text{ is primitive} \iff A^m > 0 \text{ for some } m \geq 1$$

(Note that A irreducible $\iff \forall\, i,j,\ \exists\, m_{ij}$ s.t. $[A^m]_{ij} > 0$; this difference is interesting).

Thm: $A \in M_n$ nonneg, irreducible,

$N_1, \cdots, N_n$ are nodes of Γ, and $L_i = \{k_1^{(i)}, k_2^{(i)}, \cdots\}$ is the set of lengths of directed paths in Γ started and ended at N_i.

Then

$$A \text{ primitive} \iff \gcd L_i = 1.$$

Lemma: $A \in M_n$, nonneg, irreducible,

if $A_{ii} > 0$ $\forall i$,

then $A^{n-1} > 0$.

In particular, A is primitive.

Lemma: $A \in M_n$, nonneg, primitive.

then A^k primitive, nonneg $\forall k \geq 1$.

Thm: $A \in M_n$, nonneg

If A primitive,

then $A^k > 0$, for some pos. $k \leq (n-1)n^n$.

Thm: $A \in M_n$, nonneg, primitive,

Let S be shortest cycle path in Γ.

Then $A^{n+S(n-2)} > 0$.

Dec. 3

Even if A nonneg, irreducible, the scaled limit $\lim_{n\to\infty}(\rho(A)^{-1}A)^n$ may not exist. say $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Thm: $A \in M_n$, $A \geq 0$, irreducible, $n \geq 2$, let x, y be right/left Perron vectors, then $\lim_{N\to\infty} \frac{1}{N}\sum_{m=1}^{N}(\rho(A)^{-1}A)^m = xy^T$.

Moreover, $\exists\, C = C(A) > 0$, finite, s.t. $\left\| \frac{1}{N}\sum_{m=1}^{N}(\rho(A)^{-1}A)^m - xy^T \right\| \leq \frac{C}{N}$, $\forall N$

Def: $A \in M_n$, nonneg, is stochastic if $Ae = e$ $(e = [1, \cdots, 1]^T)$,

$A \in M_n$, nonneg, is column stochastic if $e^T A = e^T$.

$A \in M_n$, nonneg, is doubly stochastic if A and A^T stochastic

Note: Stochastic matices form a compact, convex set; doubly stochastic matrices are also compact, convex.

Thm: (Birkoff) $A \in M_n$ doubly stochastic iff $\exists$ perm matrices $P_1, \cdots, P_N$, and $t_1, \cdots, t_N \in \mathbb{R}_{>0}$ $(N \leq n^2 - n + 1)$ s.t. $\sum t_i = 1$, $A = t_1 P_1 + \cdots + t_N P_N$.

Corol: The max (resp. min) of a convex (resp. concave) $\mathbb{R}$-valued fn on doubly stochastic matrices is attained at a permutation matrix.

Def: $A \in M_n$, nonneg, is <u>doubly substochastic</u> if $Ae \le e$, $e^T A \le e^T$.

Lemma: $A \in M_n$, doubly stochastic, then $\exists$ doubly stochastic S, s.t. $A \le S$.

Thm: (von Neumann). Let ordered singular values of $A, B \in M_n$ be

$$\sigma_1(A) \ge \cdots \ge \sigma_n(A) \quad \& \quad \sigma_1(B) \ge \cdots \ge \sigma_n(B).$$

Then $$\mathrm{Re}\{\mathrm{tr}(AB)\} \le \sum_{i=1}^{n} \sigma_i(A)\,\sigma_i(B)$$

Card Shuffling (洗牌)

(Gilbert - Shannon - Reeds)

Cut cards	Drop cards
$\frac{1}{2^n}\binom{n}{c}$	$\frac{L}{L+R}$
(prob. cut cards of top)	(prob drop from left).

$Q_2(\sigma)$ = prob. get order σ from single shuffle of initial deck.
(by cutting into 2 decks)

$$Q_2^{*k}(\sigma) = \sum_{\tau} Q_2(\tau)\, Q_2^{*(k-1)}(\sigma\tau^{-1})$$

(τ: intermediate state). (k: steps of shuffling).