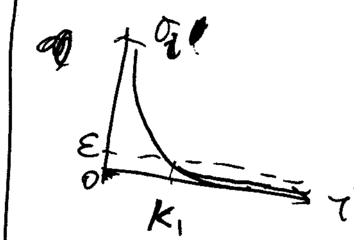
# Project: Application of SVD to signal processing.

1. Matrix theory $\longleftrightarrow$ Image processing.

| | |
|---|---|
| $A_{m\times n}$. | grayscale image. |
| $A_{i,j}$ | pixel |
| $0 \sim 255$ | black $\sim$ white |
| $R_{m\times n}, G_{m\times n}, B_{m\times n}$ | color image (RGB). |

8 bits
$2^8 = 256$

Assumption: In general, an actual image is far from full rank. (effective rank)




2. $\underset{\text{output}}{M} = \underset{\text{input}}{L} + \underset{\text{random matrix}}{N}$

Def: white Gaussian noise is a matrix, with each entry a realization of a Gaussian r.v., and the entries are mutally independent.

$\mathrm{rank}(L) = k_1$, $\mathrm{rank}(M) = k > k_1$

3. Denoise — PSNR $\uparrow$

$\uparrow$

recover $L = V\Sigma W^*$

$\uparrow$

Maximum Likelyhood estimation.

$\uparrow$ (Gauss-Markov) 1°

Nearest rank $k$ approximation.

Given $M$, solve for $L$ s.t. $M$ occurs with max prob.

Nearest rank $k_1$ estimation:

$M \in M_{m,n}$, $\operatorname{rank}(M) = k$,

Find $L \in M_{m,n}$, $\operatorname{rank}(L) = k_1 < k$, s.t.

$$L = \arg\min_L \|M - L\|_2$$

Answer: $L = V E W^*$

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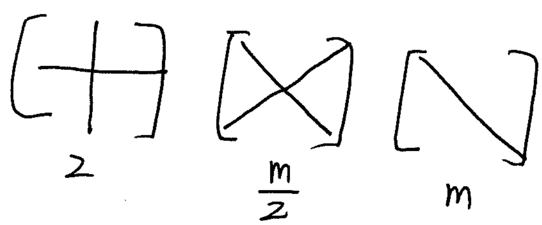
with $\begin{cases} M = V\,\Sigma(M)\,W^* \\ E \text{ takes the first } k_1 \text{ singular values of } \Sigma(M) \end{cases}$

Note: 1° Visual simplicity does not necessarily correspond to a low rank matrix.

E.g.




2° Practically useful methods may not be mathematically rigorous, but that's the way engineers do it.

Just like the simplex method is practically faster than some algorithms with polynomial complexity.

1° Proof: ( Gaussian - Markov )

white Gaussian

$$N_{i,j} \sim N(0, \sigma^2).$$

$$M_{i,j} - L_{i,j} \sim N(0, \sigma^2)$$

$M_{i,j}$: random variable. $L_{i,j}$: fixed

$$f(M_{i,j} - L_{i,j}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(M_{i,j}-L_{i,j})^2}{2\sigma^2}}$$

Likelyhood function ; maximum likelyhood estimation problem:

$$L = \arg\max_L \prod_{i=1}^{m} \prod_{j=1}^{n} f(M_{i,j} - L_{i,j})$$

$$= \arg\max_L \log \prod_{i=1}^{m} \prod_{j=1}^{n} f(M_{i,j} - L_{i,j})$$

$$= \arg\max_L \sum_{i=1}^{m} \sum_{j=1}^{n} \log f(M_{i,j} - L_{i,j})$$

$$= \arg\max_L \sum_{i=1}^{m} \sum_{j=1}^{n} \left[ -\frac{1}{2\sigma^2}(M_{i,j} - L_{i,j})^2 + \log\frac{1}{\sqrt{2\pi\sigma^2}} \right]$$

$$= \arg\min_L \sum_{i=1}^{m} \sum_{j=1}^{n} (M_{i,j} - L_{i,j})^2$$

$$= \arg\min_L \| M - L \|_2$$

$\square$

(2°)

Proof: Recall: $A, B \in M_{m,n}$, then

$\forall$ unitarily invariant norm $\|\cdot\|$ on $M_{m,n}$,

$$\|A - B\| \geq \|\Sigma(A) - \Sigma(B)\|$$

$$\therefore \|M - L\|_2 \geq \|\Sigma(M) - \Sigma(L)\|_2$$

$$= g(\sigma_1(M) - \sigma_1(L), \cdots, \sigma_{k_1}(M) - \sigma_{k_1}(L), \sigma_{k_1+1}(M), \cdots, \sigma_k(M), 0, \cdots, 0)$$

$$\geq g(0, \cdots, 0, \sigma_{k_1}(M), \cdots, \sigma_k(M), 0, \cdots, 0)$$

$$= \|M - VEW^*\|$$

$\therefore$ Take $L = VEW^*$.

$\square$

Frobineous norm is unitarily invariant.

$g(\cdot)$ symmetric gauge fn

Symmetric fns are monotone norms.

Thm : Gauss-Markov Theorem.

$$\vec{y} = X_{m\times n}\vec{\theta} + \vec{\varepsilon}$$

$$y^{(i)} = \vec{x}^{(i)T}\vec{\theta} + \varepsilon^{(i)} \qquad X_{m\times n} = \begin{bmatrix} \vec{x}^{(1)T} \\ \vdots \\ \vec{x}^{(m)T} \end{bmatrix}$$

$$= \vec{\theta}^{T} \cdot \vec{x}^{(i)} + \varepsilon^{(i)}$$

$$L(\vec{\theta}) = p(\vec{y} \mid X, \vec{\theta}) = \prod_{i=1}^{m} p(y^{(i)} \mid \vec{x}^{(i)}, \vec{\theta})$$

$$= \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ - \frac{(y^{(i)} - \vec{\theta}^{T}\vec{x}^{(i)})^2}{2\sigma^2} \right\}$$

$$\operatorname{argmax}\{L(\vec{\theta})\} \iff$$

$$\operatorname{argmax}\{\ln(L(\vec{\theta}))\} \iff \min\left\{ \sum_{i=1}^{m} (y^{(i)} - \vec{\theta}^{T}\vec{x}^{(i)})^2 \right\}$$

least square problem : $\vec{y} = X\vec{\theta}$

Probabilistic. Deterministic.