Lec. 6 Inequalities of Hermitian Matrices & Properties of Positive (Semi-)Definite

Defn: A matrix B is <u>positive semi-definite</u> if $x^* B x \geq 0$, $\forall x \in \mathbb{C}^n$

Fact: B <u>posi-s-de</u> $\Leftrightarrow$ B Hermitian & $\lambda_{min} \geq 0$.

Coroll (monotonicity): $\underbrace{\lambda_k(A) \leq \lambda_k(A+B)}_{\text{If } B \text{ posi-se-de}}$ $\forall k = 1, \dots, n$.

Note: $\mathrm{Rank}(zz^*) = 1$.

Thm (Interlacing): H Hermitian, $z \in \mathbb{C}^n \setminus 0$

(a) $\lambda_k(H + zz^*) \geq \lambda_{k+1}(H)$

(b) $\lambda_{k+1}(H) \geq \lambda_{k+2}(H + zz^*)$

Note: H Hermitian $\Rightarrow$ $\mathrm{Rank}(H) = \#\text{nonzero } \lambda_i\text{'s}$.

Thm: A, B Hermitian, $\mathrm{rank}(B) \leq r$.

$\lambda_k(A+B) \geq \lambda_{k+r}(A) \geq \lambda_{k+2r}(A+B)$.

Thm (Weyl):

$$\lambda_{i+j-n}(A+B) \geq \lambda_i(A) + \lambda_j(B) \quad (i+j \geq n+1)$$

$$\lambda_i(A) + \lambda_j(B) \leq \lambda_{i+j-1}(A+B) \quad (i+j \leq n+1)$$

Defn: $\alpha, \beta \in \mathbb{R}^n$. β <u>majorizes</u> α if $\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \beta_i$ and $\forall k = 1, \dots, n-1$, $\min\left\{\sum_{j=1}^{k} \beta_{i_j} \,\middle|\, i_1 < \dots < i_k\right\} \geq \min\left\{\sum_{j=1}^{k} \alpha_{i_j} \,\middle|\, i_1 < \dots < i_k\right\}$.

Thm: H Hermitian. Then the vector of diagonal entries of H majorizes the vector of eigenvalues.

Rayleigh - Ritz

(1) $\lambda_{max} \geq \frac{x^* H x}{x^* x} \geq \lambda_{min}$, $\forall x \in \mathbb{C}^n \setminus 0$,

(2) $\max \frac{x^* H x}{x^* x} = \lambda_{max}$

(3) $\min \frac{x^* H x}{x^* x} = \lambda_{min}$

proof: $H = U \Lambda U^*$

<u>Thm</u>: H Hermitian, $(\lambda_1 \geq \cdots \geq \lambda_n)$

$$\lambda_k = \max_{w_n, \cdots, w_{k+1}} \min_{\substack{x \in \mathbb{C}^n \setminus 0 \\ x \perp w_n, \cdots, w_{k+1}}} \frac{x^* H x}{x^* x}$$

$$= \min_{w_1, \cdots, w_{k-1}} \max_{\substack{x \in \mathbb{C}^n \setminus n \\ x \perp w_1, \cdots, w_{k-1}}} \frac{x^* H x}{x^* x}$$

proof: $\{U^* x \mid x \in \mathbb{C}^n \setminus 0\} = \{y \mid y \in \mathbb{C}^n \setminus 0\}$

$$\inf_{\substack{x \in \mathbb{C}^n \setminus 0 \\ x \perp w_n, \cdots, w_{k+1}}} \frac{x^* H x}{x^* x}$$

Theorem (Weyl) A, B Hermitian.

$$\lambda_k(A) + \lambda_{min}(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_{max}(B).$$

Proof: $\lambda_k(A+B) = \min_{\{w_1, \cdots, w_{k-1}\}} \max_{x \perp \{w_1, \cdots, w_{k-1}\}} \frac{x^*(A+B)x}{x^* x}$

$$\geq \underline{\min}\ \underline{\max} \left[\frac{x^* A x}{x^* x} \right] + \lambda_{min}(B).$$

Note: Bounds are tight. Consider $B = \alpha u_k u_k^*$.

$= E\left(\left|\sum_{i=1}^{n} \bar{z}_i (X_i - \mu_i)\right|^2\right) \geq 0$, ~~posi-def~~ convex.

Prop: $x^*Ax \in \mathbb{R}, \forall x \in \mathbb{C}^n \Rightarrow A$ Hermitian.

Rmk: 1° $A \in M_n(\mathbb{R})$ and $x^*Ax > 0, \forall x \in \mathbb{R}^n \setminus 0$

$\not\Rightarrow A$ symmetric

2° A pos-def $\Rightarrow \bar{A}, A^T, A^*, A^{-1}$ pos-def.

Prop: Any principle submatrix $A(S)$ of a pos-def matrix A is pos-def.

Prop: The set of posi-def matrices forms a positive cone in $M_n(\mathbb{C})$

Prop: A, B pos-def matrices, $a, b \in \mathbb{R}_{\geq 0}$

$\Rightarrow$ $aA + bB$ pos-def.

Thm: A Hermian matrix:

pos-(semi)def $\Leftrightarrow$ all its eigenvalues are (nonneg) positive.

Prop: Let A Hermitian, $f_A(t) = \sum_{i=0}^{n} a_i t^i$, $a_0 = a_1 = \cdots = a_{n-m-1} = 0$, $a_{n-m} \neq 0$.

($\Leftarrow$ not shown) then

A pos semi-def $\Leftrightarrow$ $a_k a_{k+1} < 0$ for $k = n-m, \cdots, n-1$.

Coroli: B posi-s-de then diagonal entries ≥ 0

Thm: A, B Hermitian:

$\vec{\lambda}(A+B)$ magorizes $\vec{\lambda}(A)+\vec{\lambda}(B)$

Lec. 7 Quadratic Form & Positive (Semi-) Definite Matrices

Matrix	eigenvalue
unitary	S^1 ← (unit sphere)
Hermitian	$\mathbb{R}$.
posi-definite	$\mathbb{R}_{>0}$.

- It's Useful to reduce a complicated function at a local scale to a quadratic form.
- Remark: Higher order derivatives only affect global properties, not local ones.

Given F with stationary point at (α, β), form:

$$f(x,y) = \frac{x^2}{2}\frac{\partial^2 f}{\partial x^2}\Big|_{(\alpha,\beta)} + xy\frac{\partial^2 F}{\partial x \partial y}\Big|_{(\alpha,\beta)} + \frac{y^2}{2}\frac{\partial^2 F}{\partial y^2}\Big|_{(\alpha,\beta)} = x^T\left(\frac{\partial^2 F}{\partial x_i \partial x_j}\right)\Big|_{(\alpha,\beta)} x$$

Hessian

behaves near (0,0) the same way F behaves near (α, β). (Given $F'' \neq 0$).

Defn: positive definite (quadratic form f).

? Thm: $f = ax^2 + bxy + cy^2$ pos-def $\iff$ $a > 0$ & $ac - b^2 > 0$.

Covariance matrix: $A = [E((\bar{x}_i - \bar{\mu}_i)(x_j - \mu_j))]$

$$z^* A z = \left(\sum_{i,j=1}^{n} \bar{z}_i E((\bar{x}_i - \bar{\mu}_i)(x_j - \mu_j)) z_j\right)$$

$$= \left(\sum_{i,j=1}^{n} E(\bar{z}_i (\bar{x}_i - \bar{\mu}_i) z_j (x_j - \mu_j))\right) = E\left[\sum [\bar{z}_i (\bar{x}_i - \bar{\mu}_i)][z_j (x_j - \mu_j)]\right]$$

Cholesky decomposition:

A pos def $\Leftrightarrow$ $\exists$ non-singular lower triangular L with posi diagonal. s.t. $A = LL^*$.

If A real, L taken real.

Thm: Let $\{w_1, \cdots, w_k\} \in V$, and an inner product $\langle , \rangle$.

set $W = [w_1, \cdots, w_k] \in M_{n,k}$,

and $G \in M_k$ Gram matrix, $G_{ij} = \langle w_j, w_i \rangle$

(a) G pos semi-def

(b) G non-sing $\Leftrightarrow$ $\{w_i\}$ lin. ind.

$\left(\begin{array}{l} \langle w_j, w_i \rangle = \overline{\langle w_i, w_j \rangle} \\ \langle cx, y \rangle = c \langle x, y \rangle \end{array} \right)$

(c) $\exists$ pos. def. $A \in M_n$ st. $G = W^* A W$.

(d) rank$\{G\}$ = rank$\{W\}$ = max # lin. ind. vectors in $\{w_i\}$

~~$x^* A x = 0$, A posi def $\Rightarrow$ $x = 0$.~~

Corollary: $A \in M_n$. Then A posi-semi-def, with rank $r \le n$.

$\Leftrightarrow$ $\exists \{w_1, \cdots, w_n\} \in \mathbb{C}^n$, with exactly r lin. ind. st.

$(A)_{ij} = \langle w_j, w_i \rangle$ ← Euclidean.

Oct. 17 INEQUALITY ABOUT DETERMINANT OF POSITIVE SEMIDEFINIT

Thm: (Hadamard) A pos semi-def, then

$$\det(A) \le \prod_{i=1}^{n} A_{ii}$$

If A pos def, then equality iff A diagonal.

Thm: For any B, we have

$$|\det(B)| \le \prod_{i=1}^{n} \sqrt{\sum_{j=1}^{n} |B_{ij}|^2}$$

$$|\det(B)| \le \prod_{j=1}^{n} \sqrt{\sum_{i=1}^{n} |B_{ij}|^2}$$

When B nonsingular, equality iff row/cols orthogonal.

(proof: ① $A = BB^*$ pos def.
② volume $\le \prod_{i=1}^{n}$ length of edge.)

Thm: (Fisher's inequality)

$P = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}$ pos def, A, C square, then

$$\det(P) \le \det A \cdot \det C.$$

Thm: (Szasz's inequality)

Define $P_k(A) = \prod$ (k×k principal minors of A).

A pos def, then

$$P_{k+1}(A)^{\binom{n-1}{k}^{-1}} \le P_k(A)^{\binom{n-1}{k-1}^{-1}} \quad (k=1,\dots,n-1)$$

$P_1(A) = \prod_{i=1}^{n} A_{ii}$, $P_n(A) = \det A$.

<u>Prop</u>: A pos semi-def,

$$\alpha(A) \equiv \begin{cases} \dfrac{\det(A)}{\det(\hat{A})}, & \text{if } \hat{A} \text{ pos def} \\ 0, & \text{otherwise.} \end{cases}$$

where $\hat{A}$ remove 1st row & col of A.

$$(E_{11})_{ij} \equiv \delta_{(1,1),(i,j)}.$$

Then

$A - tE_{11}$ is pos semi-def,

$\Leftrightarrow$ $t \leq \alpha(A)$.

In ~~[illegible]~~ particular, $A - \alpha(A) E_{11}$ is pos semi-def.

$$\left(\det(A - tE_{11}) = \det(A) - t\det(\hat{A}) \right)$$

<u>Thm</u>: (Oppenheim's inequality)

A, B pos semi-def,

$$\det(A) \cdot \left(\prod_{i=1}^{n} B_{ii} \right) \leq \det(A \circ B).$$

<u>Thm</u>: (generalize $|z| \geq |Re\{z\}|$)

Let $H = \frac{1}{2}(A + A^*)$, Then

$$|\det(A)| \geq |\det(H)|$$

with equality iff A Hermitian.

Note: If the Hermitian part (H) of A is pos-def, then

~~[illegible]~~ $Re\{\lambda(A)\} > 0$.