[Random Processes]

~~Infinite numbers of r.v.'s~~

Chap. 12 Basic descriptions of random processes. (Chap 12.1)

Thm: (Kolmogorov Existence theorem)

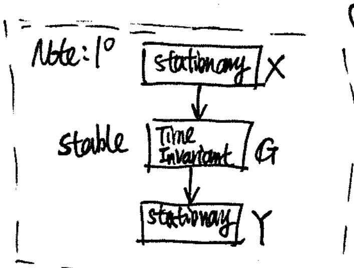
A r.p. is completely described by consistently stating the joint cdfs (or pdfs) of all possible finite collection of r.v.'s in the process.

Processes with simplified/specialized descriptions:

① iid r.v.'s with given pdf

② Gaussian process with given mean and covariance fn.

③ (Strictly) stationary r.p.: (Def. 12.1)



$$P_{X(u,t_1)\cdots X(u,t_n)}(\underline{z}) = P_{X(u,t_1+\tau)\cdots X(u,t_n+\tau)}(\underline{z})$$

$$\forall\ n\in\mathbb{Z}_+,\ t_1,\cdots,t_n\in\mathcal{J},\ \underline{z}\in\mathbb{R}^n,\ \tau\in\mathcal{J}\ (j=1,\cdots,n)$$

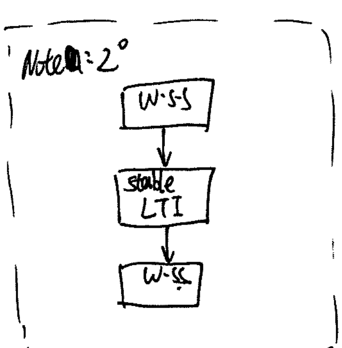
Def: wide-sense equivalent r.p.: (Eq. 12.4)

r.p.'s $X(u,t)$, $Y(u,t)$ with the same $\mathcal{U}$ and $\mathcal{J}$, and

$$\begin{cases} m_X(t) = m_Y(t) \\ K_X(t_1,t_2) = K_Y(t_1,t_2) \end{cases}$$

denote $X(u,t) \overset{w.s.}{=\!=} Y(u,t)$

④ wide-sense stationary r.p.: (w.s.s.) (Def. 12.3)



$$X(u,t\oplus\tau) \overset{w.s.}{=\!=} X(u,t) \quad \forall\ \tau,t\in\mathcal{J}$$

Note: 1° w.s.s. r.p has simplified 2nd order description:

$$\Longleftrightarrow m_X,\ K_X(\tau) = K_X(t_1+\tau,t_1)$$

$$\overset{\shortparallel}{m_X(t)}$$

(Chap. 12-1)

Goal: complete probablistic description of a r.p.

Def: (12-2) A real r.p. $X(u,t)$, is a ~~Gaussian r.p.~~ Gaussian r.p. if all finite subsets of r.v.'s in the process are jointly Gaussian, i.e.

$$p_{X(u,t_1),\cdots,X(u,t_n)}(\underline{z}) \sim \text{Gaussian},$$

$$\forall\, n \in \mathbb{Z}^+,\ t_1,\cdots,t_n \in \mathcal{T}.$$

Note: 1° pdf of a Gaussian r.v. is specified by the mean value vector and the covariance matrix, hence a real Gaussian r.p. is completely described by its mean value function and covariance function.

Def: (12.4) Two r.p.'s $X(u,t), Y(u,t)$ are independent if

$$p_{X(u,t_1),\cdots,X(u,t_m),Y(u,t_1'),\cdots,Y(u,t_n')}(\underline{z},\underline{z}')$$

$$= p_{X(u,t_1),\cdots,X(u,t_m)}(\underline{z}) \cdot p_{Y(u,t_1'),\cdots,Y(u,t_n')}(\underline{z}')$$

$$\forall\, m,n \in \mathbb{Z}^+,\ t_1,\cdots,t_m,\ t_1',\cdots,t_n' \in \mathcal{T},\ \underline{z} \in \mathbb{R}^m,\ \underline{z}' \in \mathbb{R}^n.$$

Def: (P 267) We say two r.p.'s $X(u,t)$ and $Y(u,t)$ have a property jointly, if the property applies to the collection

$C_{all} \equiv C_X \cup C_Y$, where $C_X \equiv \{X(u,t) : u \in \mathcal{U},\ t \in \mathcal{T}\}$

$C_Y \equiv \{Y(u,t) : u \in \mathcal{U},\ t \in \mathcal{T}\}$.

Note: 1° If $X(u,t)$ and $Y(u,t)$ are independent stationary/w.s.s./Gaussian r.p.'s, then they're jointly stationary/~~[illegible]~~ w.s.s./Gaussian.

r.p.'s with evolutionary descriptions:

Markov process: r.p. ~~process~~ possessing the Markov property.

Point process: r.p. for which any one realization consists of a set of isolated points either in time or geographical space, etc.

Renewal process: point process with sequence of intervals between points being iid r.v.'s.

How to tell a fn. $R(t_1,t_2)$ is a covariance fn.?

1) Hermitian; 2) posi. semi-def.

1° Function $R(t_1, t_2)$ is a covariance function, $t_1, t_2 \in \mathbb{Z}$,

$\Longleftrightarrow$ $\forall$ finite set $\mathcal{T}_F \in \mathbb{Z}$, vector $\underline{g} \in \mathbb{C}^{|\mathcal{T}_F|}$,

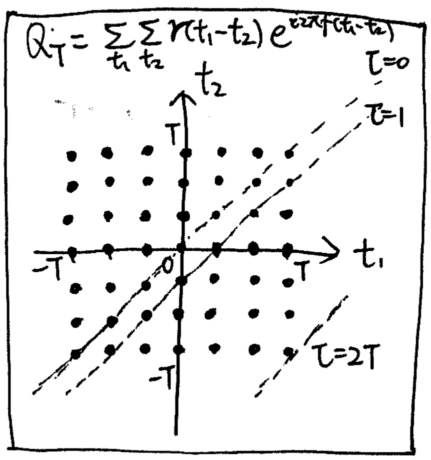
$$\sum_{t_1\in\mathcal{T}_F}\sum_{t_2\in\mathcal{T}_F} g(t_1)\, R(t_1,t_2)\, g^*(t_2) \geq 0.$$

2° For a special case (W.S.S.), $R(t_1, t_2) = r(t_1 - t_2)$, it is

$$\sum_{t_1\in\mathcal{T}_F}\sum_{t_2\in\mathcal{T}_F} g(t_1)\, r(t_1-t_2)\, g^*(t_2) \geq 0 \qquad (*)$$

$\Longrightarrow$ Suppose $\mathcal{T}_F = \{-T, \cdots, -1, 0, 1, \cdots, T\}$,

$g(t) = \hat{e}_f = e^{i2\pi f t}$



Let Q_T = LHS of $(*)$, then

$$Q_T = \sum_{\tau=-2T}^{2T} N(\tau)\, r(\tau)\, e^{i2\pi f\tau} \qquad (\tau \equiv t_1 - t_2)$$

where $N(\tau)$ is the number of points:

$$N(\tau) = 2T+1-|\tau|, \quad (|\tau| \leq 2T)$$

If $\sum_{\tau=-\infty}^{+\infty} |r(\tau)| < +\infty$, then

$$\lim_{T\to\infty} \frac{Q_T}{2T+1} = \lim_{T\to\infty} \sum_{\tau=-2T}^{2T} \left(1 - \frac{|\tau|}{2T+1}\right) r(\tau)\, e^{i2\pi f\tau}$$

$$= \sum_{\tau=-\infty}^{+\infty} r(\tau)\, e^{i2\pi f t}$$

Hence, if $\sum_{\tau=-\infty}^{+\infty} |r(\tau)| < +\infty$, then a necessary condition for $R(t_1, t_2) = r(t_1 - t_2)$ to be a covariance function is

$$\sum_{\tau\in\mathbb{Z}} r(\tau)\, e^{i2\pi f\tau} \geq 0, \quad \forall f \in [-\tfrac{1}{2}, \tfrac{1}{2})$$

Power spectral density (Chap. 14.2, 15.1)

Objective: Work in the Fourier domain to determine the effect of a LTI transformation on the input process, through which we avoid the two-dimensional integrations that must be carried out, and simplifies calculation.

In direct calculation (time domain), the output correlation fn. yields the two-dimensional integration

$$R_Y(t,t') = \iint_{-\infty}^{+\infty} h(\alpha) R_X(t-\alpha, t'-\beta) h^*(\beta)\, d\alpha\, d\beta.$$

In Fourier domain (frequency domain),

$$Y(u,f) = H(f)\, X(u,f)$$

and

$$R_Y(f,f') = H(f) R_X(f,f') H^*(f')$$

which is much simpler.

Def: Average energy $\mathcal{E}_X$ in r.p. $X(u,t)$ is

$$\mathcal{E}_X \equiv \mathbb{E} \int_{-\infty}^{+\infty} |X(u,t)|^2 dt$$

when possible, we have

$$\mathcal{E}_X = \int_{-\infty}^{+\infty} R_X(t,t)\, dt$$

$$\mathcal{E}_X = \mathbb{E} \int_{-\infty}^{+\infty} |X(u,f)|^2 df = \int_{-\infty}^{+\infty} R_X(f,f)\, df.$$

Note: 1° Here we used Plancherel's theorem (Parseval's theorem):

$$\int_{-\infty}^{+\infty} |X(u,t)|^2 dt = \int_{-\infty}^{+\infty} |X(u,f)|^2 df$$

Def: <u>(Average) energy spectral density</u> of r.p $X(u,t)$

$$E_X(f) \equiv \mathbb{E}|X(u,f)|^2 = R_X(f,f)$$

Note: Energy spectral density $E_X(f)$ is not a complete second-moment description of $X(u,t)$.

Def: <u>Truncated r.p</u> $\quad X_T(u,t) \equiv \sqcap_T(t) \cdot X(u,t) = \begin{cases} X(u,t), & |t| \le \frac{T}{2} \\ 0, & |t| > \frac{T}{2}. \end{cases}$

Note: $\mathcal{E}_{X_T} = \mathbb{E}\int_{-\frac{T}{2}}^{\frac{T}{2}} |X(u,t)|^2 \, dt$ exists for any second-order r.p $X(u,t)$ integrable in the m.s.s. over finite intervals.

~~Def: Given~~ $\cancel{\lim_{T\to+\infty} \frac{\mathcal{E}_{X_T}}{T}}$ ~~exists, the (average) power~~ $\cancel{P_X}$ ~~in r.p~~ $\cancel{X(u,t)}$ ~~is~~

$$\cancel{P_X}$$

Def: <u>(Average) power</u> P_X of r.p. $X(u,t)$ is

$$P_X \equiv \lim_{T\to+\infty} \frac{\mathcal{E}_{X_T}}{T}, \quad \text{if the limit exists.}$$

Def: <u>Power spectral density</u> (p.s.d.) $S_X(f)$ of r.p $X(u,t)$ is

$$S_X(f) \equiv \lim_{T\to+\infty} \frac{E_{X_T}(f)}{T}, \quad \text{if the limit exists.}$$

In addition, we require

$$\int_{-\infty}^{+\infty} S_X(f)\, df = \lim_{T\to+\infty} \int_{-\infty}^{+\infty} \frac{E_{X_T}(f)}{T}\, df$$

so that $\displaystyle\int_{-\infty}^{+\infty} S_X(f)\, df = P_X.$

Note: 1° The additional condition is satisfied in engineering situations in which the signal energy tends to be in the same "operating bandwidth" in all finite time intervals.

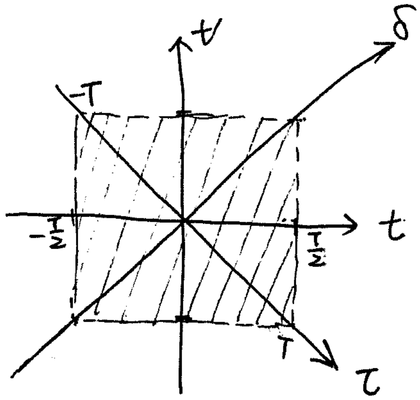
2° stationary, w.s.s., cyclo-stationary, periodic r.p.'s have p.s.d.

3° Normalized p.s.d $s_X(f) = \dfrac{S_X(f)}{R_X(0)}$, s.t. $\int_{-\infty}^{+\infty} s_X(f)\,df = 1$.

<u>Thm</u>: (Wiener - Khintchine)

If r.p. $X(u,t)$ is w.s.s., and $\int_{-\infty}^{+\infty} |R_X(\tau)|\,d\tau < \infty$, then

$$S_X(f) = \mathcal{F}\{R_X(\tau)\}$$



Proof:

$$\begin{aligned}
S_X(f) &= \lim_{T\to+\infty} \frac{1}{T} E_{X_T}(f) \\
&= \lim_{T\to+\infty} \frac{1}{T} \mathbb{E}\left| X_T(u,f)\right|^2 \\
&= \lim_{T\to+\infty} \frac{1}{T} \mathbb{E}\left| \int_{-\frac{T}{2}}^{\frac{T}{2}} X(u,t)\, e^{-i2\pi f t}\, dt\right|^2 \\
&= \lim_{T\to+\infty} \frac{1}{T} \iint_{-\frac{T}{2}}^{\frac{T}{2}} R_X(t-t')\, e^{-i2\pi f(t-t')}\, dt\, dt' \\
&= \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} R_X(\tau)\, e^{-i2\pi f\tau} \left(\int_{-T+|\tau|}^{T-|\tau|} d\delta\right) d\tau \\
&= \lim_{T\to+\infty} \int_{-T}^{T} R_X(\tau)\, e^{-i2\pi f\tau} \cdot \frac{T-|\tau|}{T}\, d\tau \\
&= \int_{-\infty}^{+\infty} R_X(\tau)\, e^{-i2\pi f\tau}\, d\tau \\
&= \mathcal{F}\{R_X(\tau)\}.
\end{aligned}$$

Note: 1° The theorem can be generalized to w.s.s. r.p. with covariance fn. $K_X(\tau)$ absolutely integrable.

$(\because S_X(f) = S_{X_0}(f) + m_X^2\,\delta_D(f) = \mathcal{F}\{R_{X_0}(\tau)\} + \mathcal{F}\{m_X^2\} = \mathcal{F}\{R_X(\tau)\})$

Properties of p.s.d. :

non-negativity – (a) $S_X(f) \geq 0 \; (\forall f \in \mathcal{F})$, $\int_{-\infty}^{+\infty} S_X(f)\, df = P_X$.

periodicity – (b) If $\mathbb{T} = \mathbb{Z}$, then $S_X(f+1) = S_X(f)$, $\forall f \in \mathbb{R}$

additivity – (c) If second-order r.p.'s $X(u,t)$, and $Y(u,t)$ are orthogonal,

$Z(u,t) = X(u,t) + Y(u,t)$,

then $S_Z(f) = S_X(f) + S_Y(f)$.

proof: (c) X, Y are orthogonal

$$\Leftrightarrow R_{XY^*}(t_1, t_2) = 0, \quad \forall t_1, t_2 \in \mathbb{R}$$

$$\Rightarrow \mathbb{E}\{X_T(u,f_1) Y_T^*(u,f_2)\} = \iint_{-\infty}^{+\infty} \mathbb{E}[X_T(u,t_1) Y_T^*(u,t_2)] e^{-i2\pi(f_1t_1 - f_2t_2)} dt_1 dt_2$$

$$= 0, \quad \forall f_1, f_2 \in \mathbb{R}$$

$$\Rightarrow E_{Z_T}(f) = \mathbb{E}\left| X_T(u,f) + Y_T(u,f)\right|^2$$

$$= \mathbb{E}|X_T(u,f)|^2 + \mathbb{E}|Y_T(u,f)|^2 + 2\mathrm{Re}\{\mathbb{E}[X_T(u,f) Y_T^*(u,f)]\}$$

$$= E_{X_T}(f) + E_{Y_T}(f)$$

$$\Rightarrow S_Z(f) = \lim_{T\to+\infty} \frac{E_{Z_T}(f)}{T} = \lim_{T\to+\infty} \frac{E_{X_T}(f)}{T} + \lim_{T\to+\infty} \frac{E_{Y_T}(f)}{T} = S_X(f) + S_Y(f)$$

symmetry – (d) If $X(u,t) \in \mathbb{R}$, then $S_X(f) = S_X(-f)$, $\forall f \in F$.

Note: 1° Prop.(b) justifies that, for random sequencies, $S_X(f)$ need only be specified for $[-\frac{1}{2}, \frac{1}{2})$

Def: Cross-power spectral density (cross-p.s.d.) $S_{XY^*}(f)$ of two jointly-WSS r.p's $X(u,t)$, $Y(u,t)$ is

$$S_{XY^*}(f) \equiv \mathcal{F}\{R_{XY^*}(\tau)\}$$

Reversely, $R_{XY^*}(\tau) = \mathcal{F}^{-1}\{S_{XY^*}(f)\}$

Note: 1° Properties:

a) Periodicity: if $\mathcal{T}=\mathbb{Z}$, then $S_{XY^*}(f+1) = S_{XY^*}(f)$, $\forall f \in \mathbb{R}$.

b) Symmetry:
- $S_{XY^*}(f) = S_{YX^*}^{*}(f)$
- If $X(u,t), Y(u,t) \in \mathbb{R}$, then $S_{XY^*}(f) = S_{XY^*}^{*}(-f)$

2° p.s.d. and cross-p.s.d. are related by

$$S_X(f) = S_{XX^*}(f)$$

3° $|S_{XY^*}(f)|^2 \leq S_X(f)\, S_Y(f)$

Dirac Delta fn's in p.s.d.

① $X(u,t)$ w.s.s., $m_X \neq 0$

$\because$ $X_0(u,t)$ and m_X are second-order and orthogonal.

$X(u,t) = X_0(u,t) + m_X$

$\therefore$ $S_X(f) = S_{X_0}(f) + S_{m_X}(f) = S_{X_0}(f) + |m_X|^2 \delta_D(f)$

② $Y(u,t) = a(u)$, $m_a = 0$, $\sigma_a^2 = \sigma^2$

$\because$ $m_Y = m_a = 0$, $R_Y(\tau) = \sigma^2$

$\therefore$ $S_Y(f) = \mathcal{F}\{R_Y(\tau)\} = \sigma^2 \delta_D(f)$

operating ~~op~~ range

?

Cannot invoke thm, $\because$ $R_Y(\tau)$ not abs. inte ~~op~~

③ $Z(u,t) = \exp\{i(2\pi f_0 t + \Theta(u))\}$, $\Theta \sim U(0, 2\pi)$.

$\because$ $m_Z = 0$, $R_Z(\tau) = e^{i2\pi f_0 \tau}$

$\therefore$ $S_Z(f) = \mathcal{F}\{R_Z(\tau)\} = \delta_D(f - f_0)$

Summary: 1° $S_X(f)$ contains $\delta_D(f)$ term $\Rightarrow$ $m_X(t) \neq 0$

2° $S_X(f)$ doesn't contain $\delta_D(f)$ term $\Rightarrow$ $m_X(t) = 0$.

Spectral representation of w.s.s. r.p.'s (Chap 14.2.3)

Def: Power spectral distribution (P.S.D.) $\mathcal{S}_X(f)$ is

$$\mathcal{S}_X(f) \equiv \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{f} E_{X_T}(f')\, df' \quad \text{, if the limit exists.}$$

When the power spectral density (p.s.d.) exists, we have

$$S_X(f) = \frac{d}{df}\mathcal{S}_X(f)$$

Def: Fourier-Stieltjes transform of r.p. $X(u,t)$, $X_I(u,f)$, is defined in the incremental form:

$$X_I(u,f_2) - X_I(u,f_1) \equiv \int_{-\infty}^{+\infty} X(u,t) \cdot \frac{e^{-i2\pi f_2 t} - e^{-i2\pi f_1 t}}{-i2\pi t}\, dt$$

Note: 1° The integrand of RHS is a finite energy process for $t \in \mathbb{R}$, so the integral converges in the m.s.s.

Thm: $X(u,t)$ is w.s.s., then

a) its Fourier-Stieltjes transform $X_I(u,f)$ exists in the m.s.s.

b) $X_I(u,f)$ is an orthogonal increment process with $R_{X_I}(f_1,f_2) = \mathcal{S}_X(\min(f_1,f_2))$.

Mathematical consideration on convergence may be useless

c) The inverse Fourier-Stieltjes transform gives $X(u,t)$:

$$X(u,t) = \int_{-\infty}^{+\infty} e^{i2\pi f t}\, dX_I(u,f)$$

Note: Transformation relations:

a) $dY_I(u,f) = H(f)\, dX_I(u,f)$

b) $d\mathcal{S}_Y(f) = |H(f)|^2\, d\mathcal{S}_X(f)$