Rigid Body,

$$\underline{r_i} \times (\underline{w} \times \underline{r_i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_i & y_i & z_i \\ (w_y z_i - w_z y_i) & (w_z x_i - w_x z_i) & (w_x y_i - w_y x_i) \end{vmatrix}$$

$$\underline{L} = \sum_{i=1}^{n} m_i \underline{r_i} \times (\underline{w} \times \underline{r_i}) \qquad (\underline{V_i} = \underline{w} \times \underline{r_i})$$

$$= \hat{i}\left[w_x \sum_{i=1}^{n} m_i (y_i^2 + z_i^2) - w_y \sum_{i=1}^{n} m_i x_i y_i - w_z \sum_{i=1}^{n} m_i x_i z_i\right]$$
$$+ \hat{j}\left[-w_x \sum_{i=1}^{n} m_i x_i y_i + w_y \sum_{i=1}^{n} m_i (x_i^2 + z_i^2) - w_z \sum_{i=1}^{n} m_i y_i z_i\right]$$
$$+ \hat{k}\left[-w_x \sum_{i=1}^{n} m_i x_i z_i - w_y \sum_{i=1}^{n} m_i y_i z_i + w_z \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)\right]$$

define $\overleftrightarrow{I} \equiv \begin{pmatrix} \sum_i m_i (y_i^2 + z_i^2), & -\sum_i m_i x_i y_i, & -\sum_i m_i x_i z_i \\ -\sum_i m_i x_i y_i, & \sum_i m_i (x_i^2 + z_i^2), & -\sum_i m_i y_i z_i \\ -\sum_i m_i x_i z_i, & -\sum_i m_i y_i z_i, & \sum_i m_i (x_i^2 + y_i^2) \end{pmatrix}$ $\left(I_{xx} \equiv \sum_i m_i (y_i^2 + z_i^2),\ I_{xy} \equiv \sum_i m_i x_i y_i, \text{ similarly} \right)$

then $\underline{L} = \overleftrightarrow{I} \cdot \underline{w}$.

~~Principle body frame~~

note : 1. if a rigid body moves with respect to a lab frame, the moment of inertia tensor calculated using a lab frame is time dependent.

2. In a body frame, the inertia tensor is time independent.

3. Always exist principle body frames with respect to which the moment of inertia tensor is diagonal.

$$T = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} \sum m_i \bar{V_i}^2 = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} \underline{w} \cdot \overleftrightarrow{I} \cdot \underline{w}$$

Euler's Equations of Motion of a Rigid Body

$$\left(\frac{d\underline{L}_{cm}}{dt}\right)_{fixed} = \left(\frac{d\underline{L}_{cm}}{dt}\right)_{rot} + \underline{\omega} \times \underline{L}_{cm}$$

where the "fixed" mean the term is calculated with respect to the lab frame, which is fixed and is an inertia frame. The "rot" mean the term is calculated with respect to the principle body frame that its origin is at the center of mass.

The fundamental equation governing the rotational part of the motion of any system, refered to an inertial coordinate system, is

$$\underline{N} = \left(\frac{d\underline{L}}{dt}\right)_{fixed}$$

the term is denoted by "fixed"
for this is an intermediate equation derived from Newton's II law.

Hence,

$$\underline{N} = \left(\frac{d\underline{L}}{dt}\right)_{rot} + \underline{\omega} \times \underline{L}$$

the equation in components along the principle axes of the body is

$$\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} = \begin{pmatrix} I_1 \dot{\omega}_1 \\ I_2 \dot{\omega}_2 \\ I_3 \dot{\omega}_3 \end{pmatrix} + \begin{pmatrix} \omega_2 \omega_3 (I_3 - I_2) \\ \omega_3 \omega_1 (I_1 - I_3) \\ \omega_1 \omega_2 (I_2 - I_1) \end{pmatrix}$$