[Simulation]

Realizable Systems (Chap. 22.4)

Def: LTI system $\mathbb{H}$ is realizable if it satisfies:

1° Stability: every bounded input produces a bounded output.

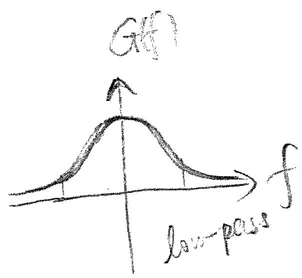
2° Causality: $\forall t \in T$, the output at time t only depends on the input at $t' \leq t$.

low-pass filter
high-pass
all-pass

Note: 1° In stable systems, bounded means
$\exists M < +\infty$, s.t. $\forall t \in T$, $|x(t)| \leq M$.

Thm: (Stability Condition)

An LTI system $\mathbb{H}$ is stable $\iff$ its impulse response $h(t)$ is absolutely integrable.



Thm: (Causality Condition)

An LTI system $\mathbb{H}$ is causal $\iff$ its impulse response $h(t) = 0$, $\forall t < 0$.

(Chap 15.4) Causality and Spectral Factorization (Chap. 22.4)

Simulating a w.s.s. Random sequence

~~Note: Since~~ $Y(u,t) = G X(u,t) = (g * X)(u,t) = (\mathcal{F}^{-1}\{G\} * X)(u,t)$

Hence ~~impulse response~~ $g(t)$, system function $G(f)$ are both alternative definitions of an LTI operator G. see "Characterizing LTI op."

$$W(u,t) \longrightarrow \boxed{\begin{matrix}\text{LTI} \\ G(f)\end{matrix}} \longrightarrow X(u,t)$$

white $\left(\begin{matrix} m_W = 0;\ R_W(\tau) = \delta(\tau) \\ \text{or } S_W(f) = 1 \end{matrix}\right)$ $\qquad m_x = 0$; $R_X(\tau)$ or $S_X(f)$

(1) Find $G(f)$.

$$\because \quad S_X(f) = S_W(f)\, |G(f)|^2$$

$$\therefore \quad G(f) = \sqrt{S_X(f)}\, e^{i\psi(f)}$$, where $\psi(f)$ is any real fn.

(2) Find $G(f)$ so that the system is causal, (i.e., $g(t) = 0$ for $t < 0$)

$$\because \quad G(f) = \sum_{\tau=-\infty}^{+\infty} g(\tau)\, e^{-i2\pi f\tau}$$

$$= \sum_{\tau=0}^{+\infty} g(\tau)\, e^{-i2\pi f\tau}$$

$$G_z(z) \equiv \sum_{\tau=0}^{+\infty} g(\tau) z^{-\tau} \qquad (z \equiv e^{i2\pi f})$$

$\because S_X(f) \geq 0$, and has period 1.

$\therefore \ln S_X(f)$ is real with period 1.

Make Fourier series expansion,

$$\ln S_X(f) = \sum_{n=-\infty}^{+\infty} a_n\, e^{-i2\pi f n}$$

where

$$a_n = \int_{-\frac{1}{2}}^{\frac{1}{2}} \ln S_X(f)\, e^{i2\pi f n}\, df$$

and $a_{-n} = a_n^*$

Write $\ln S_x(f) = \left(\frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n e^{-i2\pi f n}\right) + \left(\frac{a_0}{2} + \sum_{n=-\infty}^{-1} a_n e^{-i2\pi f n}\right)$

define $\underline{\eta(z) = \frac{a_0}{2} + \sum_{n=1}^{+\infty} a_n z^{-n}}$,

then $\ln S_x(f) = \eta(z) + \eta^*(z)$

$\because$ $\ln S_x(f) = \ln |G(f)|^2$ ~~$= \ln G(f) + \ln G^*(f)$~~

$= \ln G(f) + \ln G^*(f)$

$= \ln G(f) + [\ln G(f)]^*$

Choose $\ln G(f) = \eta(z)$, i.e., $\underline{G(f) = e^{\eta(z)}}$,

~~(then G(f) corresponds to a causal system, applying the discrete spectral factorization theorem (22-116).~~

$\because$ $\eta(z)$ is convergent, and hence analytic for all z on and outside the unit circle on z-plane

$\therefore$ $G(f)$ is <u>analytic on and outside the unit circle on z-plane</u>,

$\therefore$ The inverse z-transform of $G(f)$ is 0 for $t<0$, i.e. $g(t) = 0$ for $t<0$,

$\therefore$ $G(f)$ corresponds to a causal system. See <u>discrete spectral factorization theorem</u> (22-116) for details.

Proof details: 1° The Fourier series expansion is possible, when $\int_{-\frac{1}{2}}^{\frac{1}{2}} |\ln S_x(f)|\, df < \infty$. (discrete version of Paley-Wiener Expansion.)

2° When the filter (system) is stable, $\sum_{t=0}^{+\infty} |g(t)| < \infty$,

Hence $G_z(z) = \sum_{t=0}^{+\infty} g(t) z^{-t}$ converges on and outside the unit circle on z-plane. ($\because \sum_{t=0}^{+\infty} |g(t) z^{-t}|$ converges.)

(3) Find $G(f)$, s.t. the system is causal, given $S_X(f)$ is rational fn of z:

$$S_X(f) = C z^{\nu} \frac{\prod_{n=1}^{N}(z-u_n)}{\prod_{d=1}^{D}(z-v_d)},$$

where $u_n \neq 0$, $v_d \neq 0$, $u_n \neq v_d$, $\forall n, d$.

$\because$ $S_X(f)$ Real $\overset{(z\text{-plane})}{\Longrightarrow}$ roots (zeros)/poles occurs in conjugate reciprocal pairs, and $\nu = \frac{-N+D}{2}$

$S_X(f) \geq 0$ $\Longrightarrow$ roots (zeros)/poles on unit circle have even order,

$R_X(0) < \infty$ $\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)\,df < \infty\right)$ $\Longrightarrow$ no poles on unit circle.

$$\therefore S_X(f) = C z^{\frac{-N+D}{2}} \frac{\prod_{n=1}^{\frac{N}{2}}(z-u_n)(z-\frac{1}{u_n^*})}{\prod_{d=1}^{\frac{D}{2}}(z-v_d)(z-\frac{1}{v_d^*})}$$

where $|u_n| \leq 1$, $|v_d| < 1$

$$= C \frac{\prod_{n=1}^{\frac{N}{2}} -\frac{1}{u_n^*}}{\prod_{d=1}^{\frac{D}{2}} -\frac{1}{v_d^*}} \cdot \frac{\prod_{n=1}^{\frac{N}{2}}(1-u_n z^{-1})(1-u_n^* z)}{\prod_{d=1}^{\frac{D}{2}}(1-v_d z^{-1})(1-v_d^* z)}$$

choose $G(f) = c' e^{i\phi} \dfrac{\prod_{n=1}^{\frac{N}{2}}(1-u_n z^{-1})}{\prod_{d=1}^{\frac{D}{2}}(1-v_d z^{-1})}$

where $c' = \sqrt{C \dfrac{\prod_{n=1}^{N/2} -\frac{1}{u_n^*}}{\prod_{d=1}^{D/2} -\frac{1}{v_d^*}}}$, ϕ is any real number.

$\therefore S_X(f) = G(f)G^*(f)$
(Note: $z^{-1} = z^*$ on the unit circle)

Since $G(f)$ is analytic on and outside the unit circle in z-plane.
$\therefore$ $G(f)$ corresponds to a causal system.

If in addition, $|u_n| < 1$ $\forall n$, then $\frac{1}{G(f)}$ also corresponds to a causal system. (whitening operator) (inverse operator)

See <u>discrete rational spectral factorization theorem</u> (22.124)

Proof Details: 1° $\because$ $S_X(f)$ is real

$\therefore$ $S_X(f) = S_X^*(f)$

$$\therefore\quad C z^{\nu} \frac{\prod_{n=1}^{N}(z-u_n)}{\prod_{d=1}^{D}(z-v_d)} = C z^{-\nu} \frac{\prod_{n=1}^{N}(z^{-1}-u_n^*)}{\prod_{d=1}^{D}(z^{-1}-v_d^*)}$$

$$= C' z^{-\nu-N+D} \frac{\prod_{n=1}^{N}\left(z-\frac{1}{u_n^*}\right)}{\prod_{d=1}^{D}\left(z-\frac{1}{v_d^*}\right)}$$

$\because$ both sides have the same set of roots (and poles).

$\therefore$ $\{u_n\} = \{\frac{1}{u_n^*}\}$, $\{v_d\} = \{\frac{1}{v_d^*}\}$, $\nu = -\nu-N+D$

$\therefore$ Roots (and poles) are in conjugate reciprocal pairs, and $\nu = \frac{-N+D}{2}$.

2°. Suppose $u_1 = e^{i2\pi f_1}$ is a root of $S_X(f)$ on unit circle

write $S_X(f) = (z - e^{i2\pi f_1})^{\alpha} k(z)$,

where $k(e^{i2\pi f_1}) \neq 0$.

then
$$S_X(f_1+df) = e^{i2\pi f_1 \alpha}(e^{i2\pi df}-1)^{\alpha} k(e^{i2\pi f_1+df})$$
$$\approx e^{i2\pi f_1\alpha} k(e^{i2\pi f_1+df})(i2\pi df)^{\alpha}$$

$\because$ $S_X(f) \geq 0$

$\therefore$ α is even.

$\therefore$ Roots on unit circle in z-plane have even order.

Similarly, since $(S_X(f))^{-1} \geq 0$, poles on unit circle in z-plane also have even order.

3° Suppose $v_1 = e^{i2\pi f_1}$ is a pole of $S_X(f)$ on unit circle.

write $S_X(f) = \dfrac{1}{(z-e^{i2\pi f_1})^\alpha} k(z)$

where $k(e^{i2\pi f_1}) < \infty$, α is even. (From (2))

then

$$\int_{f_1-\varepsilon}^{f_1+\varepsilon} \frac{1}{(z-e^{i2\pi f_1})^\alpha} k(z)\, df$$

$$= \int_{-\varepsilon}^{+\varepsilon} \frac{k(e^{i2\pi(f_1+f')})}{e^{i2\pi f_1}(e^{i2\pi f'}-1)^\alpha}\, df'$$

$$\to \infty,$$

it contradicts with $R_X(0) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_X(f)\, df < \infty$.

$\therefore$ There's no pole on unit circle.

Note:

1° $H(f) = \dfrac{1}{G(f)}$ corresponds to a causal system, s.t. $W = \mathbb{H}X$.

Simulating a w.s.s. Random Waveform.

$$W(u,t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ H(f)\end{matrix}} \longrightarrow X(u,t)$$

$\begin{pmatrix} m_W=0 \\ S_W(f)=1 \end{pmatrix}$ $\qquad\qquad$ $\begin{pmatrix} m_X=0 \\ S_X(f) \end{pmatrix}$

There's no Dirac delta fn in $S_X(f)$.

(1) Find $H(f)$, s.t. the system is causal, given $S_X(f)$ is rational fn of f.

$$S_X(f) = c\,\frac{\prod_{n=1}^{N}(f-a_n)}{\prod_{d=1}^{D}(f-b_d)} \qquad (f\in\mathbb{R})$$

$\because$ $S_X(f)$ real $\Rightarrow$ roots/poles occur in conjugate pairs.

$S_X(f) \geq 0$ $\Rightarrow$ real roots have even order

$\int_{-\infty}^{+\infty} S_X(f)\,df < \infty$ $\Rightarrow$ not real poles.

$\therefore$ Can write $S_X(f) = c\cdot\left(\dfrac{\prod_{n=1}^{\frac{N}{2}}(f-a_n)}{\prod_{d=1}^{\frac{D}{2}}(f-b_d)}\right)\left(\dfrac{\prod_{n=1}^{\frac{N}{2}}(f-a_n^*)}{\prod_{d=1}^{\frac{D}{2}}(f-b_d^*)}\right)$ $(f\in\mathbb{R})$

where $\mathrm{Im}\{a_n\} \geq 0$ for $n=1,\cdots,\frac{N}{2}$,

$\mathrm{Im}\{b_d\} > 0$ for $d=1,\cdots,\frac{D}{2}$.

Choose $H(f) = \sqrt{c}\,\dfrac{\prod_{n=1}^{\frac{N}{2}}(f-a_n)}{\prod_{d=1}^{\frac{D}{2}}(f-b_d)}$ $(f\in\mathbb{R})$, so that $S_X(f)=|H(f)|^2$. $(f\in\mathbb{R})$.

Assume $\frac{D-N}{2} \geq 1$

$\because$ All the poles of $H(f)$ are in the upper-half plane,

Using inversion of Fourier transform by residues, we have ~~contour integral, it can be shown that~~ $H(f)$ corresponds to a causal system.

See continuous rational spectral factorization thm (22.136) for details.

$$\cancel{\therefore h(t) = \lim_{F\to+\infty} \hat{h}(t)}$$

$$\cancel{= \lim_{F\to+\infty} - \int_{C_L} H(f) e^{i2\pi f t} dt}$$

$$\cancel{= 0 \quad , \forall t<0}$$

Note: ~~1° For t>0, using similar technique, h(t) can be expressed by residues. (留数). (Notes to be completed.)~~

2° If $X(u,t)$ is a real r.p., poles/zeros of $S_X(f)$ are symmetric about $Im\{f\}$ and $Re\{f\}$.

However, the symmetries of $S_X(f)$ doesn't imply $X(u,t)$ to be real

3° White noise $W(u,t)$ is not w.s.s., because

$$R_W(0) = \int_{-\infty}^{+\infty} S_W(f)\,df = \int_{-\infty}^{+\infty} 1\cdot df \to +\infty,$$

which means $W(u,t)$ is not a second order process.

We can approximate the white noise w.r.t. $H(f)$ with a w.s.s. r.p., so the mathematical convergence still holds.

~~(Notes to be completed).~~

4° We don't use z-plane here.

1° For proof, together with the inversion of Fourier transform by residues, we only need to note that

$$\because |H(f)| = |H(Re^{i\theta})| \sim R^{\frac{N-D}{2}} \leq R^{-1} \quad \text{when } R \gg 1$$

$$\therefore \left| \int_{C_U} H(f) e^{i2\pi f t} df \right| \leq \int_{C_U} |H(Re^{i\theta})| \, e^{-2\pi R t \sin\theta} R \, d\theta \to 0, \text{ as } R\to\infty$$

(2) Find $H(f)$, s.t. the system is causal.

~~$\mathcal{T} = \mathbb{Z}$, $f_d \in [-\frac{1}{2}, \frac{1}{2})$; $\mathcal{T} = \mathbb{R}$, $f_c \in \mathbb{R}$~~

$\mathcal{T} = \mathbb{Z}$, $f_d \in [-\frac{1}{2}, \frac{1}{2})$	$\mathcal{T} = \mathbb{R}$, $f_c \in \mathbb{R}$
~~$f_d = \frac{1}{\pi}\arctan f_c$~~ $\xleftarrow{f_d = \frac{1}{\pi}\arctan f_c}$	$S_X(f_c)$

$$S_{X,d}(f_d) = S_X(\tan(\pi f_d))$$

get $H_d(f_d)$ from discrete spectral factorization thm $\xrightarrow{f_c = \tan(\pi f_d)}$ $H(f_c) = H_d\left(\frac{1}{\pi}\arctan f_c\right)$

Note: For Fourier series to exist, we have

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |\ln S_{X,d}(f_d)| \, df_d < +\infty$$

That is,

$$\int_{-\infty}^{+\infty} \frac{|\ln S_X(f_c)|}{\pi(1+f_c^2)} \, df_c < +\infty$$

It's called Paley-Wiener Criterion.