Part C: Special Operators

Def: <u>Adjoint operator</u> : $(Kx, y) = (x, K^*y), \ \forall x, y \in H$.

$K: H \to H$ is a continuous linear operator on a Hilbert space H.

Thm: The adjoint operator K^* is a continuous linear operator, and $\|K\| = \|K^*\|$, $(K^*)^* = K$.

Cor: $\|K^*K\| = \|KK^*\| = \|K\|^2 = \|K^*\|^2$

Thm: T is a BLT on a Hilbert space H, M is a closed linear subspace of H

M is invariant under $T \iff M^\perp$ is invariant under T^*

Def: A closed linear subspace M of a Hilbert space H <u>reduces</u> a linear transformation T, if $T(M) \subset M$, $T(M^\perp) \subset M^\perp$.

Cor: M reduces $T \iff M$ is invariant under T and T^*.

Thm: $T \in BLT[H, H]$,

1° $\overline{\mathcal{R}(T)} = \{\mathcal{N}(T^*)\}^\perp$

$\{\mathcal{N}(T)\}^\perp = \overline{\mathcal{R}(T^*)}$

2° T is unitary $\iff TT^* = T^*T = I$.

Thm: Let $P: \mathcal{D}_P \to L_2(-\infty, +\infty)$

$$Pu(x) = i\frac{du(x)}{dx}$$

$$\mathcal{D}_P = \{u \in L_2(-\infty, +\infty) : Pu \in L_2(-\infty, +\infty)\}$$

$Q: \mathcal{D}_Q \to L_2(-\infty, +\infty)$

$$Qu(x) = xu(x)$$

$$\mathcal{D}_Q = \{u \in L_2(-\infty, +\infty) : Qu \in L_2(-\infty, +\infty)\}$$

The Fourier transform F sets up a one-to-one correspondence between $\mathcal{D}_Q$ and $\mathcal{D}_P$:

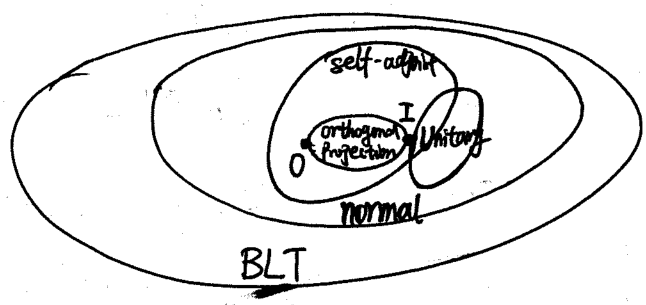
$$P = FQF^{-1}, \quad Q = F^{-1}PF.$$

Thm: Result in last theorem also holds for partial derivatives. Page 14

Def: 1° normal operator : BLT on a Hilbert space that commutes with its adjoint.

$$TT^* = T^*T$$

2° self-adjoint operator : $T = T^*$



Space of linear operators on a Hilbert space.

Thm: 1° The set of self-adjoint operators on H is closed in $Blt[H,H]$.

2° A, B are self-adjoint on H,

$$AB \text{ is self-adjoint} \iff AB = BA$$

3° $T \in BlT[H,H]$

$$T \text{ is self-adjoint} \iff (x, Tx) \in \mathbb{R}, \forall x \in H.$$

Def: Self-adjoint operator T is 1° positive if $(x, Tx) \geq 0, \forall x \in H$.

2° strictly positive if $(x, Tx) > 0, \forall x \in H, x \neq 0$.

Thm: 1° T is a self-adjoint operator,

$$\|T\| = \sup\{|(Tx, x)| : \|x\| = 1\} = \sup\{|(Tx, y)| : \|x\| = \|y\| = 1\}.$$

2° P is a continuous projection on Hilbert space :

$$\text{orthogonal} \iff \text{self-adjoint}.$$

Thm: 1° $L \in Blt[H,H]$

$$\text{normal} \iff \|L^* x\| = \|Lx\|, \forall x \in H$$

2° The set of normal operators on H is closed in $Blt[H,H]$

3° A, B are normal operators that commutes with each other's adjoint.

$\Rightarrow$ $A+B$, AB and BA are normal.

4° A, B are normal operators that commutes,

$\Rightarrow$ $A+B$, AB and BA are normal

5° L is normal $\Rightarrow$ $\|L^2\| = \|L\|^2$

Def: <u>Compact operator</u>: linear transformation between Banach spaces $L: X \to Y$
L maps the unit ball in X into a compact set in Y.

Thm: 1° Compact operators are continuous.

2° Linear operators with finite-dimensional range are compact.

3° Compact operators have almost finite-dimensional ranges:

$\forall \varepsilon > 0$, $\exists$ finite-dimensional subspace $M \subset \mathcal{R}(L)$, s.t.
$\forall x \in X$, $d(Lx, M) \le \varepsilon \|x\|$.

4° ($L: X \to Y$, a linear operator between Banach spaces).

L is compact $\Longleftrightarrow$

L maps any bounded set (in X) into a compact set (in Y).

5° A, B are compact operators $\Rightarrow$ $A+B$ is also a compact operator

6° $A, B \in Blt[X, X]$

A is compact $\Rightarrow$ AB, BA are compact.

7° The set of compact operators is closed on $Blt[X, Y]$.