[System]

Linear ~~Time~~ Translation Invariant (LTI) Operators (Chap. 22.1)

$\mathcal{L}$ is a linear space over $\mathbb{C}$,

elements $\tilde{x}$ are complex-valued fns defined on $\mathcal{T}$,

$\mathbb{H}$ denotes a linear operator on $\mathcal{L}$: $\tilde{y} = \mathbb{H}\tilde{x}$. $(\tilde{x}, \tilde{y} \in \mathcal{L})$

Def: <u>Translation operator</u> (shift operator) $\mathbb{T}_\tau$ is a linear operator shifting a fn on its domain,

$$\mathbb{T}_\tau \tilde{x} = \tilde{x}(t \oplus \tau)$$

where $\mathcal{T}$ is closed under addition $\oplus$.

Examples:

$\mathcal{T}$	$\mathcal{L}$	$\oplus$
$\mathbb{Z}_n \equiv \{0, 1, \cdots, n-1\}$	$\mathbb{C}^n$	addition modulo n
$\mathbb{Z}$	$\mathcal{L}^2(\mathbb{Z})$ ~~space of sequence~~	integer addition
$\mathbb{R}_T \equiv [0, T)$	$\mathcal{L}^2[0, T)$	addition modulo T
$\mathbb{R}$	$\mathcal{L}^2(\mathbb{R})$	addition in the real numbers

Def: ~~Linear~~ <u>Linear time invariant (LTI) operator</u> is a linear operator $\mathbb{H}$ that commutes with any translation operator on $\mathcal{T}$,

$$\mathbb{T}_\tau \mathbb{H} = \mathbb{H}\mathbb{T}_\tau, \quad \forall \tau \in \mathcal{T}$$

<u>Thm</u>: When $\mathcal{T} = \mathbb{Z}_n$, a LTI operator $\mathbb{H}$ $\Leftrightarrow$ a circulant matrix H.

proof: When $\mathcal{T} = \mathbb{Z}_n$, $\mathcal{L} = \mathbb{C}^n$.

Linear operators are matrices on $\mathbb{C}^{n\times n}$.

Let $H \in \mathbb{C}^{n\times n}$, then

H is time-invariant (*)

$\Leftrightarrow \forall \tau \in \mathbb{Z}$ $T_\tau H = H T_\tau$

where T_τ is translation operator : $T_\tau x(t) = x(t \oplus \tau)$

and $\oplus$ is addition modulo n.

$\because (T_\tau)_{ij} x_j = x_{i\oplus\tau}$

$\therefore (T_\tau)_{ij} = \delta_{(i\oplus\tau)j}$

$\therefore$ (*) $\Leftrightarrow$ $\forall \tau \in \mathbb{Z}$

$(T_\tau)_{ik} H_{kj} = H_{il} (T_\tau)_{lj}$

$\Leftrightarrow$ $\forall \tau \in \mathbb{Z}$

$\delta_{(i\oplus\tau)k} H_{kj} = H_{il} (T_\tau)_{(l\oplus\tau)j}$

$\Leftrightarrow$ $\forall \tau \in \mathbb{Z}$

$H_{(i\oplus\tau)j} = H_{i(j\oplus\tau)}$

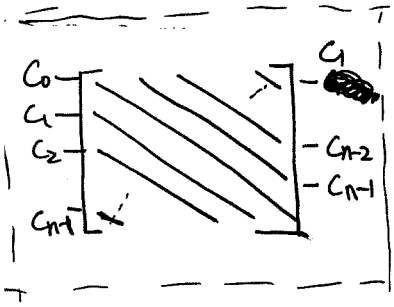
$\Leftrightarrow$ $\forall \tau \in \mathbb{Z}$, $H_{ij} = H_{(i\oplus\tau)(j\oplus\tau)}$

i.e. H only depends on $i \ominus j$,

i.e. H is circulant.

$\square$.

Circulant Matrix



Def: ~~Matrix $C = [c_{ij}]$ is a Circulant~~

Circulant matrix C is a $n \times n$ matrix, with entries c_{ij} only depend on $i \ominus j$, i.e. $(i-j) \bmod n$.

Thm: (C Complex field)

~~of eigenvectors of C~~

a) Circulant matrices have a same set of n-orthogonal eigenvectors $\{\tilde{e}_f\}$, with $\tilde{e}_f = [e^{i2\pi ft}]$, $f \in \mathcal{F} = \{\frac{\nu}{n} : \nu \in \mathbb{Z}_n\}$, $t \in \mathcal{T} = \mathbb{Z}_n$, $\mathbb{Z}_n = \{0, 1, \cdots, n-1\}$.

b) The corresponding eigen values are

$$\lambda_f = \tilde{e}_1 \cdot C \cdot \tilde{e}_f = (c_0, c_{n-1}, \cdots, c_1)\, \tilde{e}_f = \tilde{e}_f^* \cdot (c_0, c_1, \cdots, c_{n-1})^T, \quad f \in \mathcal{F}$$

c) Circulant matrices are simultaneously unitarily diagonalizable by the unitary DFT matrix, U_n.

$$C = U_n^* \Lambda_c U_n = \frac{1}{n} F_n^* \Lambda_c F_n = \frac{1}{n} E_n \Lambda_c E_n^*$$

where

$(U_n U_n^* = I)$. $U_n = \frac{1}{\sqrt{n}} F_n$, $F_n = E_n^* = [e^{-i2\pi ft}]$, $E_n = [\tilde{e}_f]$

$\Lambda_c = \mathrm{diag}\{\lambda_f\}$.

d) Circulant matrices form a commutative algebra:

$\forall A, B$ circulant, 1° $A+B$ circulant
2° AB circulant
3° $AB = BA$

e) Circulant matrices form an n-dimensional vector space.

$$C = c_0 I + c_1 P + \cdots + c_{n-1} P^{n-1}$$

where $P = \begin{pmatrix} 0 & 1 \\ I_{n-1} & 0 \end{pmatrix}$ is the "cyclic permutation" matrix.

Def: e-functions of linear operator $\mathbb{H}$ are fns $\tilde{e}$, s.t.:

$$\mathbb{H}\tilde{e} = \lambda\tilde{e}\ ,\ \text{while may not have } \tilde{e} \in \mathcal{L}.$$

Thm: (a) The complex exponential functions

$$\tilde{e}_f(t) \equiv e^{i2\pi f t}\ ,\ f \in \mathcal{F},\ t \in \mathcal{T}$$

are e-functions of shift operator $\mathbb{T}_\tau$ $(\tau \in \mathcal{T})$, with index set $\mathcal{F}$ given as:

$\mathcal{T}$	$\mathcal{F}$	Notes	$\tilde{e}_f \in \mathcal{L}$?
$\mathbb{Z}_n$	$\{\frac{\nu}{n} : \nu \in \mathbb{Z}_n\}$	$\tilde{e}_f \in \mathbb{C}^n$	✓
$\mathbb{Z}$	$[-\frac{1}{2}, \frac{1}{2})$	$\tilde{e}_f \notin \mathcal{L}^2(\mathbb{Z})$	×
$\mathbb{R}_T$	$\{\frac{\nu}{T} : \nu \in \mathbb{Z}\}$	$\tilde{e}_f \in \mathcal{L}^2[0,T)$	✓
$\mathbb{R}$	$\mathbb{R}$	$\tilde{e}_f \notin \mathcal{L}^2(\mathbb{R})$	×

proof: Case 1: $\mathcal{T} = \mathbb{R}$

$$\mathbb{T}_\tau \tilde{e}_f = e^{i2\pi f(t+\tau)} = e^{i2\pi f \tau}\, \tilde{e}_f$$

Case 2: $\mathcal{T} = \mathbb{Z}_n$

$$\begin{aligned}
\mathbb{T}_\tau \tilde{e}_f &= e^{i2\pi f(t \oplus \tau)} \\
&= e^{i2\pi f b} \qquad (t+\tau = an + b,\ b \in \mathbb{Z}_n,\ a \in \{0,1\}) \\
&= e^{i2\pi f(\tau - an)}\, \tilde{e}_f \\
&= e^{i2\pi f}\, \tilde{e}_f\ ,\ \text{when } e^{-i2\pi f n} = 1 \text{ for } f \in \mathcal{F}.
\end{aligned}$$

Given $\mathcal{F} = \{\frac{\nu}{n} : \nu \in \mathbb{Z}_n\}$, the equation is true.

Case 3: $\mathcal{T} = \mathbb{Z}$

$$\mathbb{I}_\tau \tilde{e}_f = e^{i2\pi f(t+\tau)}$$
$$= e^{i2\pi f \tau} \tilde{e}_f$$

If $e^{i2\pi f_1 \tau} = e^{i2\pi f_2 \tau}$, then $(f_1 - f_2)$ is an integer multiple of $\frac{1}{\tau}$.

$\therefore \mathcal{F} = [-|\frac{1}{2\tau}|, |\frac{1}{2\tau}|)$, $\tau \in \mathbb{Z}$.

Case 4: $\mathcal{T} = \mathbb{R}_T$

$$\mathbb{I}_\tau \tilde{e}_f = e^{i2\pi f(t \oplus \tau)}$$
$$= e^{i2\pi f b} \qquad (t+\tau = aT + b,\ b \in \mathbb{R}_T,\ a \in \{0,1\})$$
$$= e^{-i2\pi f a T} e^{i2\pi f \tau} \tilde{e}_f$$
$$= e^{i2\pi f \tau} \tilde{e}_f \quad \text{when } e^{-i2\pi f T} = 1$$

Given $\mathcal{F} = \{\frac{\nu}{T} : \nu \in \mathbb{Z}\}$, the equation is satisfied. $\square$.

Thm: (b) e-functions of shift operator $\mathbb{I}_\tau$ are e-functions of LTI operators.

proof: $\mathbb{I}_\tau \mathbb{H} \tilde{e}_f = \mathbb{H} \mathbb{I}_\tau \tilde{e}_f$
$$= \mathbb{H} e^{i2\pi f \tau} \tilde{e}_f$$
$$= e^{i2\pi f \tau} \mathbb{H} \tilde{e}_f$$

$\therefore (\mathbb{H}\tilde{e}_f)(t \oplus \tau) = e^{i2\pi f \tau} (\mathbb{H}\tilde{e}_f)(t)$

$\therefore (\mathbb{H}\tilde{e}_f)(\tau) = e^{i2\pi f \tau} (\mathbb{H}\tilde{e}_f)(0)$

$\therefore \mathbb{H}\tilde{e}_f = H(f)\, \tilde{e}_f$

where $H(f) = (\mathbb{H}\tilde{e}_f)(0)$.

$\square$.

e-functions and DFT

<u>Def</u>: $\tilde{e}_f \equiv [\, e^{i2\pi f t} \,]$ $\quad f \in \mathcal{F} = \{ \frac{\nu}{n} : \nu \in \mathbb{Z}_n \}$, $t \in \mathcal{T} = \mathbb{Z}_n$

$\mathbb{Z}_n = \{0, \cdots, n-1\}$.

<u>Thm</u>: a) e-functions are orthogonal.

$$\tilde{e}_f \cdot \tilde{e}_g^* = 0 \text{ , if } f \neq g$$

b) $|\tilde{e}_f| = \sqrt{n}$.

<u>Def</u>: $E_n \equiv [\, \tilde{e}_f \,]$, $f \in \mathcal{F}$

<u>Thm</u>: a) $E_n E_n^* = nI$; b) $E_n = E_n^T$; c) E_n is a Van Der Monde matrix

<u>Def</u>: DFT matrix is $F_n \equiv E_n^* = [\, e^{-i2\pi f t} \,]$

Characterizing LTI Transformations

$$x(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ G\end{matrix}} \longrightarrow y(t)$$

1° impulse input

$$\delta(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ G\end{matrix}} \longrightarrow g(t) \qquad \boxed{g(t):\ \text{impulse response}}$$

Note: $\delta(t)$ is kronecker Delta fn. for discrete time, and Dirac Delta fn. for continuous time.

2° arbitrary input

$$y(t) = (g * x)(t) \implies Y(f) = G(f)\,X(f)$$

$$\left(\begin{aligned} &\because x(t) = (\delta * x)(t) \\ &\therefore y(t) = G\,x(t) = \left[G(\delta * x)\right](t) = (g * x)(t) \end{aligned} \right)$$

3° e-function input

$$\tilde{e}_f(t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ G\end{matrix}} \longrightarrow G(f)\,\tilde{e}_f(t)$$

$$G(f) \equiv \mathcal{F}\{g(t)\} \qquad \boxed{G(f):\ \text{system function}}$$

$$\left(\because G(f)\,\tilde{e}_f(t) = (g * \tilde{e}_f)(t) = \mathcal{F}\{g(t)\}\,\tilde{e}_f(t) \right)$$

Note: System function is the e-value of the corresponding LTI system.

Note: 1° Since $y(t) = (g * x)(t)$, $Y(f) = G(f)\,X(f)$, impulse response $g(t)$ and system function $G(f)$ both completely characterize an LTI system G, apart from specifying the relation:

$$y(t) = G\,x(t).$$