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Transformation

Thm: $X \sim F_X(x)$, $Y = g(X)$, Supports $\mathcal{X}$ and $\mathcal{Y}$ are defined as

$$\mathcal{X} = \{x : f_X(x) > 0\}, \quad \mathcal{Y} = \{y : y = g(x),\ x \in \mathcal{X}\}$$

1° If $g(\cdot)$ is increasing on $\mathcal{X}$, then $Y \sim F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$

2° If $g(\cdot)$ is decreasing on $\mathcal{X}$, and X is a continuous r.v, then

$Y \sim 1 - F_X(g^{-1}(y))$ for $y \in \mathcal{Y}$

Thm: (Probability integral transformation)

1° $X \sim F_X(x)$, which is continuous, and $Y = F_X(X)$,

then $Y \sim U(0,1)$.

2° $U \sim U[0,1]$, $\forall$ cdf $F(\cdot)$, let $X = F^{-1}(U)$,

then $X \sim F(x)$

Main proof of 1°: For $0 < y < 1$,

$$\begin{aligned} P(Y \le y) &= P(F_X(X) \le y) \\ &= P(F_X^{-1}(F_X(X)) \le F_X^{-1}(y)) \quad (\because F_X^{-1} \text{ increasing}) \\ &= P(X \le F_X^{-1}(y)) \\ &= F_X(F_X^{-1}(y)) \\ &= y \end{aligned}$$

and $P(Y \le y) = \begin{cases} 1, & y \ge 1 \\ 0, & y \le 0 \end{cases}$

Special examples:

1° If $X_1 \sim \Gamma(\alpha_1, \beta)$, $X_2 \sim \Gamma(\alpha_2, \beta)$ and $X_1 \perp\!\!\!\perp X_2$.

then $X_1 + X_2 \sim \Gamma(\alpha_1 + \alpha_2, \beta)$

2° If $Z \sim N(0,1)$, then $Z^2 \sim \Gamma(\frac{1}{2}, 2) = \chi^2(1)$;

If $Z_1, \cdots, Z_n$ iid. $N(0,1)$, then $Z_1^2 + \cdots + Z_n^2 \sim \Gamma(\frac{n}{2}, 2) = \chi^2(n)$.

3° If $X_1, \cdots, X_{n+1}$ iid. Exponential (1), $S_k = \sum_{i=1}^{k} X_i \quad (k = 1, \cdots, n+1)$,

then $\left(\frac{S_1}{S_{n+1}}, \cdots, \frac{S_n}{S_{n+1}} \right) \sim (U_{(1)}, \cdots, U_{(n)})$, where $U_i \sim U[0,1]$.

proof: $(X_1, \cdots, X_{n+1}) \sim e^{-x_1} \cdots e^{-x_{n+1}} \mathbb{1}(x_1 > 0, \cdots, x_{n+1} > 0)$

$\underline{S}$ is a linear, nonsingular transformation of $\underline{X}$, and

$$\left| J_{\underline{X}}(\underline{S}) \right| = 1$$

then $(S_1, \cdots, S_{n+1}) \sim e^{-S_{n+1}} \mathbb{1}(0 < S_1 < S_2 < \cdots < S_{n+1})$

Let $(W_1, \cdots, W_n, W_{n+1}) = \left(\frac{S_1}{S_{n+1}}, \cdots, \frac{S_n}{S_{n+1}}, S_{n+1} \right)$,

then $\underline{W}$ is a 1-1 transformation with

$$\left| J_{\underline{S}}(\underline{W}) \right| = \frac{1}{S_{n+1}^n}$$

$\therefore (W_1, \cdots, W_{n+1}) \sim W_{n+1}^n e^{-W_{n+1}} \mathbb{1}(0 < W_1 < \cdots < W_n < 1,\ W_{n+1} > 0)$

$$= \left[n!\, \mathbb{1}(0 < W_1 < \cdots < W_n < 1) \right] \cdot \left[\frac{W_{n+1}^n}{n!} e^{-W_{n+1}} \mathbb{1}(W_{n+1} > 0) \right]$$

$\therefore (W_1, \cdots, W_n) \perp\!\!\!\perp W_{n+1}$, and $(W_1, \cdots, W_n) \sim n!\, \mathbb{1}(0 < W_1 < \cdots < W_n < 1)$,

$\sim (U_{(1)}, \cdots, U_{(n)})$ $\square$.

4° Given $U_{(r)} \sim \text{Beta}(r, n+1-r)$, we get $U_{(b)} - U_{(a)} \sim U_{(b-a)}$ $(b > a)$

proof: $U_{(b)} - U_{(a)} \overset{d}{=} \frac{S_b - S_a}{S_{n+1}} = \frac{X_{a+1} + \cdots + X_b}{S_{n+1}}$

$\overset{d}{=} \frac{X_1 + \cdots + X_{b-a}}{S_{n+1}}$ (exchangable)

$\sim U_{(b-a)}$

$\sim \text{Beta}(b-a, n+1-b+a)$ $\square$.

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Function of two random variables $Z = g(X, Y)$

1° Direct method:

Transformation ~~Theory~~

$$F_Z(z) = P_R\{Z \le z\}$$
$$= P_R\{g(X,Y) \le z\}$$
$$= P_R\{(X,Y) \in G_z\}$$
$$= \iint_{G_z} f_{X,Y}(x,y)\,dx\,dy$$

$$f_Z(z) = \frac{d}{dz} F_Z(z)$$

Eg.: $Z = X+Y$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\,dy$$

If $X \perp\!\!\!\perp Y$, $f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y)\, f_Y(y)\,dy$

$$= (f_X * f_Y)(z)$$

Characteristic functions:

$$\Phi_Z(\omega) = \Phi_X(\omega) \cdot \Phi_Y(\omega)$$

Eg 2: $\underline{Y} = A\underline{X} + \underline{b}$

$$\Phi_{\underline{Y}}(\underline{\nu}) = \mathbb{E}\, e^{i\underline{\nu}^T\cdot(A\underline{X}+\underline{b})}$$
$$= e^{i\underline{\nu}^T\cdot \underline{b}}\, \mathbb{E}\, e^{i(A^T\underline{\nu})^T\cdot \underline{X}}$$
$$= e^{i\underline{\nu}^T\cdot \underline{b}}\, \Phi_{\underline{X}}(A^T\underline{\nu})$$

1° X, Y exponential $(\lambda_X > \lambda_Y)$

$$f_Z(z) = \frac{\lambda_X \lambda_Y}{\lambda_X - \lambda_Y}\left[e^{-\lambda_Y z} - e^{-\lambda_X z}\right] u(z)$$

2° X, Y Gaussian

$$f_Z(z) = N(z;\, m_X + m_Y,\, \sigma_X^2 + \sigma_Y^2)$$

3° If X, Y Poisson, $Z \sim \text{Poisson}(\theta_X + \theta_Y)$

Eg: $Z = \max\{X, Y\}$, $f_Z(z) = \frac{d}{dz} F_{X,Y}(z,z)$

2°. Conditional method

1° Find $f_{Z|X}$ from $f_{Y|X}$ and $g_X(Y) = g(X, Y)$

2° $f_Z(z) = \int_{-\infty}^{+\infty} f_{Z|X}(z|x)\, f_X(x)\, dx$

3° Transformation of r.v.'s:

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} g_1(X_1, X_2) \\ g_2(X_1, X_2) \end{pmatrix}$$

$$f_{\underline{Y}}(\underline{y}) = \sum_{\underline{x}_i : \underline{g}(\underline{x}_i) = \underline{y}} \frac{f_{\underline{X}}(\underline{x}_i)}{|J_{\underline{g}}(\underline{x}_i)|}$$

Eg.: 1° $\underline{Y} = A\underline{X} + \underline{b}$, A invertible.
(Affine transformation)

$$f_{\underline{Y}}(\underline{y}) = \frac{f_{\underline{X}}(A^{-1}(\underline{y} - \underline{b}))}{|\det(A)|}$$

If $\underline{X}$ jointly Gaussian, then $\underline{Y}$ also jointly Gaussian!

with $\begin{cases} \underline{m}_y = A\underline{m}_x + \underline{b} \\ K_y = A K_x A^T \end{cases}$

If $X \sim \text{beta}(\alpha, \beta)$,
$Y \sim \text{beta}(\alpha+\beta, \gamma)$,
and $X \perp\!\!\!\perp Y$, then.
$XY \sim \text{beta}(\alpha, \beta+\gamma)$.

Eg.: 2° $\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} R \\ \Theta \end{pmatrix}$

$$f_{R,\Theta}(r, \theta) = r\, f_{X,Y}(r\cos\theta, r\sin\theta)$$

If X, Y are independent, standard Gaussian, then $\frac{X}{Y} \sim$ Cauchy (1)