III. Variational Methods

Systems with several degrees-of-freedom

virtual displacement : δq

Holonomic constraint: $f(x, y, z, t) = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z = 0 . \qquad (1)$$

1° Principle of Virtual Work ~~$\vec{R}\cdot\delta\vec{r} = 0$ restraint force~~

For particle in equilibrium: $\underbrace{\vec{R}}_{\text{restraint force}} = \underbrace{\vec{F}}_{\text{applied force}} + \underbrace{\vec{f}}_{\text{constraint force}} = 0 \qquad \Rightarrow \vec{R}\cdot\delta\vec{r} = 0$

From (1), we have ?

$$\vec{f}\cdot\delta\vec{r} = 0 \qquad \Rightarrow \vec{F}\cdot\delta\vec{r} = 0$$

For a system of N particles,

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta\vec{r}_i = 0 \qquad \text{—— principle of virtual work}$$

2° D'Alembert's principle.

Since $\vec{F} + \vec{f} \underbrace{- \dot{\vec{p}}}_{\text{inertia force}} = 0$;

for a system of N particles,

$$\sum_{i=1}^{N} (\vec{F}_i - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0$$

3° Hamilton's principle:

Of all variations, the dynamical path of a system will lead to a stationary value of action integral: $\int_{t_1}^{t_2} (T + \underbrace{W}_{\text{virtual work of external force}})\, dt$

4° Lagrange's equations

- n DOF holonomic system with independent generalized coordinates q_k.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad \text{general force:} \left(Q_i \equiv \sum_{j=1}^{m} \vec{F}_j \cdot \frac{\partial \vec{r}_j}{\partial q_i} \right)$$

For conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

Variational Principles in Mechanics

1. Hamilton's Principle: $\delta S = 0$ $\quad\vdots\quad$ $\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W_{nc}})\,dt = 0.$

   S is the action integral: $S = \int_{t_1}^{t_2} L\,dt$

   L is the Lagrangian: $L = T - V$

   W_{nc} is the work done by non-conservative force.

2. Lagrange's Equation

   - (of the second kind)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \qquad (j = 1, 2, \cdots, s)$$

T is the kinetic energy of the system: $T = \sum_{i=1}^{n} \frac{1}{2} m_i \underline{v_i}^2$

$q_1, \cdots, q_s$ are independent generalized coordinates

Q_j are generalized forces: $Q_j = \sum_{i=1}^{n} \underline{F_i} \cdot \frac{\partial \underline{r_i}}{\partial q_j}$

$\underline{F_i}$ is the total force except constraint force on the i-th particle.

This is a system of n particles under k ideal constrains.

s is the degree of freedom of the system: $s = 3n - k$.

(For conservative system)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad (j = 1, 2, \cdots, s)$$

All $\underline{F_i}$ are conservative and have a total potential V:

$V = V(\underline{r}_1, \cdots, \underline{r}_n) = V(q_1, \cdots, q_s, t)$

• (of the first kind)

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}_j}\right) - \dfrac{\partial T}{\partial x_j} = Q_j + \displaystyle\sum_{i=1}^{k} \lambda_i \dfrac{\partial f_i}{\partial x_j} & (j=1,2,\cdots,3n) \\ \cancel{\displaystyle\sum_{j=1}^{3n} \dfrac{\partial f_i}{\partial x_j}\delta x_j = 0 \quad (i=1,2,\cdots,k)} & \\ f_i(x_1, \cdots, x_{3n}, t) = 0 & (i=1,2,\cdots,k) \end{cases}$$

$x_1, \cdots, x_{3n}$ are dependent coordinates.

$$Q_j = \sum_{i=1}^{n} \underline{F}_i \cdot \frac{\partial r_i}{\partial x_j}$$

f_i are constraints.

3. Lagrange-d'Alembert's Principle

$$\sum_{i=1}^{n} (\underline{F}_i - m_i \underline{a}_i) \cdot \delta \underline{r}_i = 0$$

This is a system of n particles, $\underline{F}_i$ is the total nonconstraint force on the i-th particle.

4. Principle of Virtual Work. ~~(Virtual Displacement)~~

$$\sum_{i=1}^{N} \vec{F}_i \cdot \delta \vec{r}_i = 0 \iff \text{Equilibrium Conditions: } Q_j = 0 \quad (j=1,2,\cdots,s)$$

All the constraints should be ideal.

$4 \rightarrow 3 \searrow 2$

$1 \nearrow 2$

$2 \iff$ Newton's second law

$2 \iff 5$

- Ideal constraints:

$$\sum_{i=1}^{n} \vec{N_i} \cdot \delta \vec{r_i} = 0$$

$\vec{N_i}$ is the constraint force on the i-th particle.

5、Hamilton's Equations

$$\begin{cases} \dot{q}_j = \dfrac{\partial H}{\partial p_j} \\ \dot{p}_j = -\dfrac{\partial H}{\partial q_j} \end{cases} \qquad (j = 1, 2, \cdots, s)$$

p_i is the generalized momentum: $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$

H is Hamiltonian: $H = \sum_{i=1}^{s} p_i \dot{q}_i - L = H(\underline{q}, \underline{p}, t)$

6、最小总势能原理：(材料力学、弹性力学)

$$\frac{\partial \Pi}{\partial q_i} = 0 \quad \Big| \quad \delta \Pi = 0$$

总势能 $\Pi = V - W_{nc}$；W_{nc}为非保守力做功；V为应变能.

7、最小总余能原理：(材料力学、弹性力学)

$$\frac{\partial \Psi}{\partial Q_i} = 0 \quad \Big| \quad \delta K = 0$$

总余能 $\Psi = \Phi - q_s Q_s$

余能 $\Phi = \underline{Q} \cdot \underline{q} - V$

$1 \longrightarrow 6$

$\quad \longrightarrow 7$ (对偶形式)

8、虚功原理（结构(静)力学）

处于平衡的变形体系，对于任何虚位移，外力虚功等于变形虚功.

$$\delta V = \delta W_{外}$$

$$1 \to 8$$