W.s.s. – LTI Relationship (Chap. 15.2)

$$X(u,t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ \mathbb{G}\end{matrix}} \longrightarrow V(u,t)$$

$$Y(u,t) \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ \mathbb{H}\end{matrix}} \longrightarrow W(u,t)$$

Suppose $X(u,t)$ and $Y(u,t)$ are jointly w.s.s., defined on the same $\mathcal{U}, \mathcal{T}$.

$\mathbb{G}$, $\mathbb{H}$ are stable LTI operators.

Then $V(u,t)$ and $W(u,t)$ exist in the m.s.s.:

$$V(u,t) = (g * X)(u,t), \quad W(u,t) = (h * Y)(u,t)$$

Since $V(u,t)$, $W(u,t)$ are respectively w.s.s., and their cross-correlation fn.

$$R_{VW^*}(t_1, t_2) = \mathbb{E}\left[(g*X)(u,t_1) \cdot (h*Y)^*(u,t_2)\right]$$

$$= (g * R_{XY^*} * h^*)(t_1,t_2)$$

$$= (g * R_{XY^*} * h^*)(\tau)$$

$\therefore$ $V(u,t)$, $W(u,t)$ are jointly w.s.s.

Using cross-power spectral density, we have the following w.s.s.-LTI relationship

$$\begin{cases} S_V(f) = |G(f)|^2 S_X(f), \quad m_V = G(0)\, m_X \\ S_{VX^*}(f) = G(f)\, S_X(f) \\ S_{VW^*}(f) = G(f)\, S_{XY^*}(f)\, H^*(f) \end{cases}$$

Note: 1° ("Convolution thm")

$$\mathcal{F}\{g * R_{XY^*} * h^*\} = \mathcal{F}\{g\} \cdot \mathcal{F}\{R_{XY^*}\} \cdot \mathcal{F}\{h\}^*$$