An Example- 2 Closed Subspaces with Zero Aperture (angle)

Definition The *angle between subspaces* M and N of an inner product space is defined to be

$$\Theta(M,N) \doteq \inf\,((x,y),$$

where the infimum is taken over all $x \in M$ with $\|x\| = 1$ and $y \in N$ with $\|y\| = 1$.

Definition Define subspaces of ℓ_2 as

$$M = \text{cl } V(e_1, e_3, e_5, \ldots) \quad \text{and} \quad N = \text{cl } V(z_1, z_2, \ldots), \tag{1}$$

where" cl" denotes "closure", e_j denotes the "standard" unit vector $e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j}, \ldots)$ where δ_{ij} is the Kronecker delta and

$$z_n = \sqrt{1 - \frac{1}{n^2}}\; e_{2n-1} + \frac{1}{n^2} e_{2n}.$$

We will show,

(1) $M \cap N = \{0\}$, but $\Theta(M,N) = 0$,

(2) $M + N \neq \ell_2$,

(3) $\overline{M+N} = \ell_2$.

Note: This shows, in particular, the sum of two closed "disjoint" sub spaces need not be closed.

(a) It is easily seen that $\|z_n\| = 1$ and $(z_m, z_n) = \delta_{mn}$.

(b) If $y \in M \cap N$ and $y \neq 0$, then

$$y = \sum_1^\infty a_j e_{2j-1} = \sum_1^\infty b_j z_j.$$

But $(e_{2k-1}, z_j) = \sqrt{1 - \frac{1}{n^2}}\; \delta_{kj}$, and therefore,

$$|(y,y)| = \left|(\sum_1^\infty a_j e_{2j-1}, \sum_1^\infty b_j z_j)\right| = \left|\sum_1^\infty a_j b_j \sqrt{1 - \frac{1}{j^2}}\right| \leq \sum_1^\infty |a_j||b_j| \sqrt{1 - \frac{1}{j^2}} < \sum_1^\infty |a_j||b_j| \leq \|y\|,$$

a contradiction. Therefore $y = 0$.

(c) To see $M + N \neq \ell_2$, let $y = \sum_1^\infty \frac{1}{n} e_{2n}$ and suppose $y = x + z$, with $x \in M$ and $z \in N$. Then

$$x = \sum_1^\infty a_n e_{2n-1} \quad \text{and} \quad y = \sum_1^\infty b_n \left(\sqrt{1 - \frac{1}{n^2}}\; e_{2n-1} + \frac{1}{n^2} e_{2n} \right),$$

and therefore

$$\sum_1^\infty \frac{1}{n} e_{2n} = y = x + z = \sum_1^\infty a_n e_{2n-1} + b_n \sqrt{1 - \frac{1}{n^2}}\; e_{2n-1} + \frac{b_n}{n} e_{2n}.$$

Since $e_{2n} \perp e_{2n-1}$, we have $\dfrac{1}{n} = \dfrac{b_n}{n}$ and therefore $b_n = 1$ and the series for z diverges. Thus $y \notin M + N$.

(d) We finally see $y \in \overline{M+N}$ since

$$y = \sum_1^\infty \frac{1}{n} e_{2n} = \lim_{K\to\infty} \sum_1^K \left(z_n - \sqrt{1 - \frac{1}{n^2}} \right).$$